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**Representations by linear combinations of triangular numbers**

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**Abstract**

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. For $a_1,\ldots,a_k,n\in\mathbb{Z}^+$ let $t(a_1,\ldots,a_k;n)$ be the number of representations of $n$ by $a_1x_1(x_1+1)/2+a_2x_2(x_2+1)/2+\cdots+a_kx_k(x_k+1)/2$, and let $N(a_1,\ldots,a_k;n)$ be the number of representations of $n$ by $a_1x_1^2+a_2x_2^2+\cdots+a_kx_k^2$, where $x_1,x_2,\ldots,x_k\in\mathbb{Z}$. In this paper, using theta function identities we establish seventeen transformation formulas for $t(a_1,\ldots,a_k;n)$, and reveal many relations between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$, where $a,b,c,d\in\mathbb{Z}^+$.

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# 1. Introduction

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. For $a,b,c,d\in\mathbb{Z}^+$ and $n\in\{0,1,2,\ldots\}$ let $N(a,b,c,d;n)$ be the number of representations of $n$ by $ax^2+by^2+cz^2+dw^2$, where $x,y,z,w\in\mathbb{Z}$. In 1828 Jacobi showed that

$$N(1,1,1,1;n)=8\sum_{d\mid n,4\nmid d}d. \tag{1.1}$$

From 1859 to 1866 Liouville made about 90 conjectures on $N(a,b,c,d;n)$. During 2007-2009 A. Alaca, S. Alaca, M.F. Lemire and K.S. Williams published a series of papers on the evaluation of $N(a,b,c,d;n)$. See their typical papers [2-6] and Cooper's survey paper [9].

The numbers $x(x+1)/2$ $(x\in\mathbb{Z})$ are called triangular numbers. Let $\mathbb{N}=\{0,1,2,\ldots\}$. For $a_1,a_2,\ldots,a_k\in\mathbb{Z}^+$ $(k\ge 2)$ and $n\in\mathbb{N}$ set

$$\begin{aligned}&N(a_1,a_2,\ldots,a_k;n)=\big|\{(x_1,\ldots,x_k)\in\mathbb{Z}^k\mid n\ =a_1x_1^2+a_2x_2^2+\cdots+a_kx_k^2\}\big|,\\&t(a_1,a_2,\ldots,a_k;n)\\&=\Big|\Big\{(x_1,\ldots,x_k)\in\mathbb{Z}^k\mid n\ =a_1\frac{x_1(x_1+1)}{2}+a_2\frac{x_2(x_2+1)}{2}+\cdots+a_k\frac{x_k(x_k+1)}{2}\Big\}\Big|.\end{aligned}$$

In 2005 Adiga, Cooper and Han [1] showed that for $a_1+\cdots+a_k\le 7$,

$$t(a_1,\ldots,a_k;n)=\frac{2}{2+\binom{i_1}{4}+\binom{i_1}{2}i_2+i_1i_3}N(a_1,\ldots,a_k;8n+a_1+\cdots+a_k), \tag{1.2}$$

where $i_j$ is the number of elements in $\{a_1,\ldots,a_k\}$ which are equal to $j$.

In [10] the author proved that for $a,b,n\in\mathbb{Z}^+$ with $2\nmid a$,

$$\begin{aligned}&t(a,a,2a,4b;4n+3a)=4t(a,2a,4a,b;n),\quad t(a,a,6a,4b;4n+3a)=2t(a,a,6a,b;n),\\&t(a,a,8a,2b;2n)=t(a,2a,2a,b;n),\quad t(a,a,8a,2b;2n+a)=2t(a,4a,4a,b;n).\end{aligned}$$

In [12] the author stated 13 transformation formulas for $t(a_1,\ldots,a_k;n)$. In particular,

$$t(a,a,2b_1,\ldots,2b_r;2n+a)=2t(a,4a,b_1,\ldots,b_r;n), \tag{1.3}$$

$$t(a,3a,4b_1,\ldots,4b_r;4n+3a)=2t(3a,4a,b_1,\ldots,b_r;n), \tag{1.4}$$

$$t(a,3a,4b_1,\ldots,4b_r;4n+6a)=2t(a,12a,b_1,\ldots,b_r;n), \tag{1.5}$$

where $a,b_1,\ldots,b_r\in\mathbb{Z}^+$ with $2\nmid a$.

Section 2 is devoted to preliminary facts. In Section 3, using theta function identities we establish seventeen transformation formulas for $t(a_1,\ldots,a_k;n)$, where $a_1,\ldots,a_k,n\in\mathbb{Z}^+$. See Theorems 3.1 and 3.2. In Section 4 we prove that for odd positive integers $a$ and $b$,

$$\begin{aligned}&t(a,3a,3a,2b;n)=N(3a,3a,4a,2b;8n+7a+2b),\\&t(a,a,2a,b;n)=2N(a,4a,8a,b;8n+4a+b)\quad\text{for } a\equiv -b \pmod 4,\\&t(a,6a,6a,b;n)=\frac{1}{3}N(a,3a,48a,4b;32n+52a+4b)\quad\text{for } a\equiv b \pmod 4.\end{aligned}$$

In Section 5 we prove some special relations between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$ under certain congruence conditions, and pose 7 challenging conjectures based on calculations with Maple.

# 2. Preliminary facts

Since $\frac{(-x-1)(-x)}{2}=\frac{x(x+1)}{2}$ and $8\cdot\frac{x(x+1)}{2}=(2x+1)^2-1$, it is easy to see that for $a_1,\ldots,a_k\in\mathbb{Z}^+$ and $n\in\mathbb{N}$,

$$\begin{aligned}&t(a_1,a_2,\ldots,a_k;n)\\&=2^k\Big|\Big\{(x_1,\ldots,x_k)\in\mathbb{N}^k\ \Big|\ n\ =a_1\frac{x_1(x_1+1)}{2}+a_2\frac{x_2(x_2+1)}{2}+\cdots+a_k\frac{x_k(x_k+1)}{2}\Big\}\Big|\\&=\big|\{(x_1,\ldots,x_k)\in\mathbb{Z}^k\ \big|\ 8n+a_1+\cdots+a_k=a_1x_1^2+a_2x_2^2+\cdots+a_kx_k^2,\ 2\nmid x_1\cdots x_k\}\big|\\&=2^k\big|\{(x_1,\ldots,x_k)\in\mathbb{N}^k\ \big|\ 8n+a_1+\cdots+a_k=a_1x_1^2+a_2x_2^2+\cdots+a_kx_k^2,\ 2\nmid x_1\cdots x_k\}\big|.\end{aligned} \tag{2.1}$$

Let $\varphi(q)$ and $\psi(q)$ be Ramanujan's theta functions defined by

$$\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}=1+2\sum_{n=1}^{\infty}q^{n^2}\quad\text{and}\quad\psi(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}\ (|q|<1).$$

It is easy to see that for $a_1,\dots,a_k\in\mathbb{Z}^+$ and $|q|<1$,

$$\sum_{n=0}^{\infty} t(a_1,\dots,a_k;n)q^n = 2^k\psi(q^{a_1})\cdots\psi(q^{a_k}), \tag{2.2}$$
$$\sum_{n=0}^{\infty} N(a_1,\dots,a_k;n)q^n = \varphi(q^{a_1})\cdots\varphi(q^{a_k}). \tag{2.3}$$

There are many identities involving $\varphi(q)$ and $\psi(q)$. From [7, Lemma 4.1] or [8] we know that for $|q|<1$,

$$\psi(q)^2 = \varphi(q)\psi(q^2), \tag{2.4}$$
$$\varphi(q) = \varphi(q^4)+2q\psi(q^8) = \varphi(q^{16})+2q^4\psi(q^{32})+2q\psi(q^8), \tag{2.5}$$
$$\varphi(q)^2 = \varphi(q^2)^2+4q\psi(q^4)^2 = \varphi(q^4)^2+4q^2\psi(q^8)^2+4q\psi(q^4)^2, \tag{2.6}$$
$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4)+q\varphi(q^2)\psi(q^{12}). \tag{2.7}$$

By (2.4)-(2.6) or [10, Lemma 2.4],

$$\varphi(q)^2 = \varphi(q^8)^2+4q^4\psi(q^{16})^2+4q^2\psi(q^8)^2+4q\varphi(q^{16})\psi(q^8)+8q^5\psi(q^8)\psi(q^{32}). \tag{2.8}$$

By (2.5) and (2.7),

$$\begin{aligned}\varphi(q)\varphi(q^3) &= (\varphi(q^4)+2q\psi(q^8))(\varphi(q^{12})+2q^3\psi(q^{24}))\\ &= \varphi(q^4)\varphi(q^{12})+4q^4\psi(q^8)\psi(q^{24})+2q(\varphi(q^{12})\psi(q^8)+q^2\varphi(q^4)\psi(q^{24}))\\ &= \varphi(q^4)\varphi(q^{12})+4q^4\psi(q^8)\psi(q^{24})+2q\psi(q^2)\psi(q^6).\end{aligned} \tag{2.9}$$

By [8, p.315],

$$\psi(q)\psi(q^7) = \psi(q^8)\varphi(q^{28})+q\psi(q^2)\psi(q^{14})+q^6\varphi(q^4)\psi(q^{56}). \tag{2.10}$$

Thus,

$$\psi(q^2)\psi(q^{14}) = \psi(q^{16})\varphi(q^{56})+q^2\psi(q^4)\psi(q^{28})+q^{12}\varphi(q^8)\psi(q^{112}).$$

This together with (2.10) gives

$$\begin{aligned}\psi(q)\psi(q^7) &= \psi(q^8)\varphi(q^{28})+q^6\varphi(q^4)\psi(q^{56})+q\psi(q^{16})\varphi(q^{56})\\ &\quad +q^3\psi(q^4)\psi(q^{28})+q^{13}\varphi(q^8)\psi(q^{112}).\end{aligned} \tag{2.11}$$

From [8, p.377] we know that for $|q|<1$,

$$\psi(q^3)\psi(q^5) = \varphi(q^{60})\psi(q^8)+q^3\psi(q^2)\psi(q^{30})+q^{14}\varphi(q^4)\psi(q^{120}). \tag{2.12}$$

By [13, (2.10)],

$$\psi(q)\psi(q^{15}) = \psi(q^6)\psi(q^{10})+q\varphi(q^{20})\psi(q^{24})+q^3\varphi(q^{12})\psi(q^{40}). \tag{2.13}$$

Thus,

$$\psi(q^2)\psi(q^{30}) = \psi(q^{12})\psi(q^{20})+q^2\varphi(q^{40})\psi(q^{48})+q^6\varphi(q^{24})\psi(q^{80}).$$

This together with (2.12) yields

$$\psi(q^3)\psi(q^5) = \varphi(q^{60})\psi(q^8) + q^{14}\varphi(q^4)\psi(q^{120}) + q^3\psi(q^{12})\psi(q^{20}) + q^5\varphi(q^{40})\psi(q^{48}) + q^9\varphi(q^{24})\psi(q^{80}). \tag{2.14}$$

By (2.12),

$$\psi(q^6)\psi(q^{10}) = \varphi(q^{120})\psi(q^{16}) + q^6\psi(q^4)\psi(q^{60}) + q^{28}\varphi(q^8)\psi(q^{240}).$$

Combining this with (2.13) gives

$$\psi(q)\psi(q^{15}) = \varphi(q^{120})\psi(q^{16}) + q^{28}\varphi(q^8)\psi(q^{240}) + q^6\psi(q^4)\psi(q^{60}) + q\varphi(q^{20})\psi(q^{24}) + q^3\varphi(q^{12})\psi(q^{40}). \tag{2.15}$$

# 3. Transformation formulas for $t(a_1, \ldots, a_k; n)$

In this section we present new transformation formulas for $t(a_1, \ldots, a_k; n)$, where $a_1, \ldots, a_k, n \in \mathbb{Z}^+$.

**Theorem 3.1.** *Suppose $a, b_1, \ldots, b_r \in \mathbb{Z}^+$ with $2 \nmid a$. For $n \in \mathbb{Z}^+$ we have*

$$t(a, 7a, 2b_1, \ldots, 2b_r; 2n + a) = t(a, 7a, b_1, \ldots, b_r; n), \tag{3.1}$$

$$t(a, 7a, 8b_1, \ldots, 8b_r; 8n + 10a) = 2t(4a, 7a, b_1, \ldots, b_r; n), \tag{3.2}$$

$$t(a, 7a, 8b_1, \ldots, 8b_r; 8n + 28a) = 2t(a, 28a, b_1, \ldots, b_r; n), \tag{3.3}$$

$$t(a, 7a, 8a, 4b_1, \ldots, 4b_r; 4n + 6a) = t(a, a, 14a, b_1, \ldots, b_r; n), \tag{3.4}$$

$$t(a, 7a, 56a, 4b_1, \ldots, 4b_r; 4n) = t(2a, 7a, 7a, b_1, \ldots, b_r; n). \tag{3.5}$$

Proof. Using (2.2) and (2.10) we see that

$$\sum_{n=0}^{\infty} t(a, 7a, 2b_1, \ldots, 2b_r; n)q^n = 2^{r+2}\psi(q^a)\psi(q^{7a})\psi(q^{2b_1}) \cdots \psi(q^{2b_r})$$
$$= 2^{r+2}(\psi(q^{8a})\varphi(q^{28a}) + q^{6a}\varphi(q^{4a})\psi(q^{56a}) + q^a\psi(q^{2a})\psi(q^{14a}))\psi(q^{2b_1}) \cdots \psi(q^{2b_r})$$

and so

$$\sum_{n=0}^{\infty} t(a, 7a, 2b_1, \ldots, 2b_r; 2n + a)q^{2n+a} = 2^{r+2}q^a\psi(q^{2a})\psi(q^{14a})\psi(q^{2b_1}) \cdots \psi(q^{2b_r}).$$

This yields

$$\sum_{n=0}^{\infty} t(a, 7a, 2b_1, \ldots, 2b_r; 2n + a)q^n = 2^{r+2}\psi(q^a)\psi(q^{7a})\psi(q^{b_1}) \cdots \psi(q^{b_r}).$$

Hence (3.1) is true. Applying (2.11) we see that

$$\sum_{n=0}^{\infty} t(a, 7a, 4b_1, \ldots, 4b_r; n)q^n = 2^{r+2}\psi(q^a)\psi(q^{7a})\psi(q^{4b_1}) \cdots \psi(q^{4b_r})$$
$$= 2^{r+2}(\psi(q^{8a})\varphi(q^{28a}) + q^{6a}\varphi(q^{4a})\psi(q^{56a}) + q^a\psi(q^{16a})\varphi(q^{56a}) + q^{3a}\psi(q^{4a})\psi(q^{28a}) + q^{13a}\varphi(q^{8a})\psi(q^{112a}))\psi(q^{4b_1}) \cdots \psi(q^{4b_r}).$$

Thus,

$$\sum_{n=0}^{\infty} t(a,7a,4b_1,\cdots,4b_r;4n)q^{4n} = 2^{r+2}\psi(q^{8a})\varphi(q^{28a})\psi(q^{4b_1})\cdots\psi(q^{4b_r}),$$
$$\sum_{n=0}^{\infty} t(a,7a,4b_1,\ldots,4b_r;4n+6a)q^{4n+6a} = 2^{r+2}q^{6a}\varphi(q^{4a})\psi(q^{56a})\psi(q^{4b_1})\cdots\psi(q^{4b_r})$$

and so

$$\sum_{n=0}^{\infty} t(a,7a,4b_1,\ldots,4b_r;4n)q^{n} = 2^{r+2}\psi(q^{2a})\varphi(q^{7a})\psi(q^{b_1})\cdots\psi(q^{b_r}), \tag{3.6}$$
$$\sum_{n=0}^{\infty} t(a,7a,4b_1,\ldots,4b_r;4n+6a)q^{n} = 2^{r+2}\varphi(q^{a})\psi(q^{14a})\psi(q^{b_1})\cdots\psi(q^{b_r}). \tag{3.7}$$

By (3.7) and (2.5),

$$\sum_{n=0}^{\infty} t(a,7a,8b_1,\ldots,8b_r;4n+6a)q^{n} = 2^{r+2}(\varphi(q^{4a})+2q^{a}\psi(q^{8a}))\psi(q^{14a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}).$$

Therefore

$$\sum_{n=0}^{\infty} t(a,7a,8b_1,\ldots,8b_r;4(2n+a)+6a)q^{2n+a} = 2^{r+3}q^{a}\psi(q^{8a})\psi(q^{14a})\psi(q^{2b_1})\cdots\psi(q^{2b_r})$$

and so

$$\sum_{n=0}^{\infty} t(a,7a,8b_1,\ldots,8b_r;4(2n+a)+6a)q^{n} = 2^{r+3}\psi(q^{4a})\psi(q^{7a})\psi(q^{b_1})\cdots\psi(q^{b_r}).$$

This yields (3.2). Similarly, from (3.6) and (2.5),

$$\sum_{n=0}^{\infty} t(a,7a,8b_1,\cdots,8b_r;4n)q^{n} = 2^{r+2}(\varphi(q^{28a})+2q^{7a}\psi(q^{56a}))\psi(q^{2a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}).$$

Therefore,

$$\sum_{n=0}^{\infty} t(a,7a,8b_1,\ldots,8b_r;4(2n+7a))q^{2n+7a} = 2^{r+3}q^{7a}\psi(q^{56a})\psi(q^{2a})\psi(q^{2b_1})\ldots\psi(q^{2b_r})$$

and so

$$\sum_{n=0}^{\infty} t(a,7a,8b_1,\ldots,8b_r;4(2n+7a))q^{n} = 2^{r+3}\psi(q^{a})\psi(q^{28a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

which gives (3.3). Recall that $\varphi(q)\psi(q^2)=\psi(q)^2$. From (3.6) and (3.7),

$$\sum_{n=0}^{\infty} t(a,7a,56a,4b_1,\ldots,4b_r;4n)q^{n}$$

$$= 2^{r+3}\psi(q^{2a})\varphi(q^{7a})\psi(q^{14a})\psi(q^{b_1})\cdots\psi(q^{b_r}) = 2^{r+3}\psi(q^{2a})\psi(q^{7a})^2\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

$$\sum_{n=0}^{\infty} t(a,7a,8a,4b_1,\ldots,4b_r;4n+6a)q^n$$

$$= 2^{r+3}\varphi(q^{a})\psi(q^{2a})\psi(q^{14a})\psi(q^{b_1})\cdots\psi(q^{b_r}) = 2^{r+3}\psi(q^{a})^2\psi(q^{14a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

which yields (3.4) and (3.5). The proof is now complete.

**Theorem 3.2.** *Suppose $a, b_1, \ldots, b_r \in \mathbb{Z}^+$ with $2 \nmid a$. For $n \in \mathbb{Z}^+$ we have*

$$t(3a,5a,2b_1,\ldots,2b_r;2n+3a) = t(a,15a,b_1,\ldots,b_r;n), \tag{3.8}$$
$$t(a,15a,2b_1,\ldots,2b_r;2n) = t(3a,5a,b_1,\ldots,b_r;n), \tag{3.9}$$
$$t(3a,5a,4b_1,\ldots,4b_r;4n+3a) = t(3a,5a,b_1,\ldots,b_r;n), \tag{3.10}$$
$$t(a,15a,4b_1,\ldots,4b_r;4n+6a) = t(a,15a,b_1,\ldots,b_r;n), \tag{3.11}$$
$$t(3a,5a,8b_1,\ldots,8b_r;8n+18a) = 2t(4a,15a,b_1,\ldots,b_r;n), \tag{3.12}$$
$$t(3a,5a,8b_1,\ldots,8b_r;8n+60a) = 2t(a,60a,b_1,\ldots,b_r;n), \tag{3.13}$$
$$t(3a,5a,8a,4b_1,\ldots,4b_r;4n+14a) = t(a,a,30a,b_1,\ldots,b_r;n), \tag{3.14}$$
$$t(3a,5a,120a,4b_1,\ldots,4b_r;4n) = t(2a,15a,15a,b_1,\ldots,b_r;n), \tag{3.15}$$
$$t(a,15a,8b_1,\ldots,8b_r;8n+15a) = 2t(5a,12a,b_1,\ldots,b_r;n), \tag{3.16}$$
$$t(a,15a,8b_1,\ldots,8b_r;8n+21a) = 2t(3a,20a,b_1,\ldots,b_r;n), \tag{3.17}$$
$$t(a,15a,24a,4b_1,\ldots,4b_r;4n+3a) = t(3a,3a,10a,b_1,\ldots,b_r;n), \tag{3.18}$$
$$t(a,15a,40a,4b_1,\ldots,4b_r;4n+a) = t(5a,5a,6a,b_1,\ldots,b_r;n). \tag{3.19}$$

Proof. Using (2.12) and (2.13) we see that

$$\sum_{n=0}^{\infty} t(3a,5a,2b_1,\ldots,2b_r;n)q^n = 2^{r+2}\psi(q^{3a})\psi(q^{5a})\psi(q^{2b_1})\cdots\psi(q^{2b_r})$$
$$= 2^{r+2}(\varphi(q^{60a})\psi(q^{8a}) + q^{14a}\varphi(q^{4a})\psi(q^{120a}) + q^{3a}\psi(q^{2a})\psi(q^{30a}))\psi(q^{2b_1})\cdots\psi(q^{2b_r}),$$
$$\sum_{n=0}^{\infty} t(a,15a,2b_1,\ldots,2b_r;n)q^n = 2^{r+2}\psi(q^{a})\psi(q^{15a})\psi(q^{2b_1})\cdots\psi(q^{2b_r})$$
$$= 2^{r+2}(\psi(q^{6a})\psi(q^{10a}) + q^{a}\varphi(q^{20a})\psi(q^{24a}) + q^{3a}\varphi(q^{12a})\psi(q^{40a}))\psi(q^{2b_1})\cdots\psi(q^{2b_r}).$$

Hence

$$\sum_{n=0}^{\infty} t(3a,5a,2b_1,\ldots,2b_r;2n+3a)q^{2n+3a} = 2^{r+2}q^{3a}\psi(q^{2a})\psi(q^{30a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}),$$
$$\sum_{n=0}^{\infty} t(a,15a,2b_1,\ldots,2b_r;2n)q^{2n} = 2^{r+2}\psi(q^{6a})\psi(q^{10a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}).$$

It then follows that

$$\sum_{n=0}^{\infty} t(3a,5a,2b_1,\ldots,2b_r;2n+3a)q^{n} = 2^{r+2}\psi(q^{a})\psi(q^{15a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$
$$\sum_{n=0}^{\infty} t(a,15a,2b_1,\ldots,2b_r;2n)q^{n} = 2^{r+2}\psi(q^{3a})\psi(q^{5a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

which implies (3.8) and (3.9). By (3.8) and (3.9),

$$t(3a,5a,4b_1,\dots,4b_r;4n+3a)=t(a,15a,2b_1,\dots,2b_r;2n)=t(3a,5a,b_1,\dots,b_r;n),$$
$$t(a,15a,4b_1,\dots,4b_r;4n+6a)=t(3a,5a,2b_1,\dots,2b_r;2n+3a)=t(a,15a,b_1,\dots,b_r;n).$$

Thus, (3.10) and (3.11) hold. Appealing to (2.14),

$$\begin{aligned}\sum_{n=0}^{\infty} t(3a,5a,4b_1,\dots,4b_r;n)q^n &= 2^{r+2}\psi(q^{3a})\psi(q^{5a})\psi(q^{4b_1})\cdots\psi(q^{4b_r})\\ &= 2^{r+2}(\varphi(q^{60a})\psi(q^{8a})+q^{14a}\varphi(q^{4a})\psi(q^{120a})+q^{3a}\psi(q^{12a})\psi(q^{20a})\\ &\quad +q^{5a}\varphi(q^{40a})\psi(q^{48a})+q^{9a}\varphi(q^{24a})\psi(q^{80a}))\psi(q^{4b_1})\cdots\psi(q^{4b_r}).\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} t(3a,5a,4b_1,\dots,4b_r;4n)q^{4n}=2^{r+2}\varphi(q^{60a})\psi(q^{8a})\psi(q^{4b_1})\cdots\psi(q^{4b_r}),$$
$$\sum_{n=0}^{\infty} t(3a,5a,4b_1,\dots,4b_r;4n+14a)q^{4n+14a}=2^{r+2}q^{14a}\varphi(q^{4a})\psi(q^{120a})\psi(q^{4b_1})\cdots\psi(q^{4b_r}),$$

which yields

$$\sum_{n=0}^{\infty} t(3a,5a,4b_1,\dots,4b_r;4n)q^n=2^{r+2}\varphi(q^{15a})\psi(q^{2a})\psi(q^{b_1})\cdots\psi(q^{b_r}), \tag{3.20}$$
$$\sum_{n=0}^{\infty} t(3a,5a,4b_1,\dots,4b_r;4n+14a)q^n=2^{r+2}\varphi(q^{a})\psi(q^{30a})\psi(q^{b_1})\cdots\psi(q^{b_r}). \tag{3.21}$$

From (3.21), (3.20) and (2.5) we see that

$$\sum_{n=0}^{\infty} t(3a,5a,8b_1,\dots,8b_r;4n+14a)q^n=2^{r+2}(\varphi(q^{4a})+2q^{a}\psi(q^{8a}))\psi(q^{30a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}),$$
$$\sum_{n=0}^{\infty} t(3a,5a,8b_1,\dots,8b_r;4n)q^n=2^{r+2}(\varphi(q^{60a})+2q^{15a}\psi(q^{120a}))\psi(q^{2a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}).$$

Thus,

$$\sum_{n=0}^{\infty} t(3a,5a,8b_1,\dots,8b_r;4(2n+a)+14a)q^{2n+a}=2^{r+3}q^{a}\psi(q^{8a})\psi(q^{30a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}),$$
$$\sum_{n=0}^{\infty} t(3a,5a,8b_1,\dots,8b_r;4(2n+15a))q^{2n+15a}=2^{r+3}q^{15a}\psi(q^{120a})\psi(q^{2a})\psi(q^{2b_1})\cdots\psi(q^{2b_r})$$

and so

$$\sum_{n=0}^{\infty} t(3a,5a,8b_1,\dots,8b_r;4(2n+a)+14a)q^n=2^{r+3}\psi(q^{4a})\psi(q^{15a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

$$\sum_{n=0}^{\infty} t(3a,5a,8b_1,\ldots,8b_r;4(2n+15a))q^n = 2^{r+3}\psi(q^a)\psi(q^{60a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

which yields (3.12) and (3.13). By (3.21) and (2.4),

$$\sum_{n=0}^{\infty} t(3a,5a,8a,4b_1,\ldots,4b_r;4n+14a)q^n$$
$$= 2^{r+3}\varphi(q^a)\psi(q^{30a})\psi(q^{2a})\psi(q^{b_1})\cdots\psi(q^{b_r}) = 2^{r+3}\psi(q^a)^2\psi(q^{30a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

which implies (3.14). By (3.20) and (2.4),

$$\sum_{n=0}^{\infty} t(3a,5a,120a,4b_1,\ldots,4b_r;4n)q^n$$
$$= 2^{r+3}\varphi(q^{15a})\psi(q^{2a})\psi(q^{30a})\psi(q^{b_1})\cdots\psi(q^{b_r}) = 2^{r+3}\psi(q^{2a})\psi(q^{15a})^2\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

which yields (3.15). Using (2.15) we see that

$$\begin{aligned}\sum_{n=0}^{\infty} t(a,15a,4b_1,\ldots,4b_r;n)q^n &= 2^{r+2}\psi(q^a)\psi(q^{15a})\psi(q^{4b_1})\cdots\psi(q^{4b_r})\\ &= 2^{r+2}(\varphi(q^{120a})\psi(q^{16a}) + q^{28a}\varphi(q^{8a})\psi(q^{240a}) + q^{6a}\psi(q^{4a})\psi(q^{60a})\\ &\quad + q^a\varphi(q^{20a})\psi(q^{24a}) + q^{3a}\varphi(q^{12a})\psi(q^{40a}))\psi(q^{4b_1})\cdots\psi(q^{4b_r}).\end{aligned}$$

From this it follows that

$$\sum_{n=0}^{\infty} t(a,15a,4b_1,\ldots,4b_r;4n+a)q^{4n+a} = 2^{r+2}q^a\varphi(q^{20a})\psi(q^{24a})\psi(q^{4b_1})\cdots\psi(q^{4b_r}),$$
$$\sum_{n=0}^{\infty} t(a,15a,4b_1,\ldots,4b_r;4n+3a)q^{4n+3a} = 2^{r+2}q^{3a}\varphi(q^{12a})\psi(q^{40a})\psi(q^{4b_1})\cdots\psi(q^{4b_r})$$

and so

$$\sum_{n=0}^{\infty} t(a,15a,4b_1,\ldots,4b_r;4n+a)q^n = 2^{r+2}\varphi(q^{5a})\psi(q^{6a})\psi(q^{b_1})\cdots\psi(q^{b_r}), \tag{3.22}$$
$$\sum_{n=0}^{\infty} t(a,15a,4b_1,\ldots,4b_r;4n+3a)q^n = 2^{r+2}\varphi(q^{3a})\psi(q^{10a})\psi(q^{b_1})\cdots\psi(q^{b_r}). \tag{3.23}$$

Hence, applying (2.5) we get

$$\sum_{n=0}^{\infty} t(a,15a,8b_1,\ldots,8b_r;4n+a)q^n = 2^{r+2}(\varphi(q^{20a}) + 2q^{5a}\psi(q^{40a}))\psi(q^{6a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}),$$
$$\sum_{n=0}^{\infty} t(a,15a,8b_1,\ldots,8b_r;4n+3a)q^n = 2^{r+2}(\varphi(q^{12a}) + 2q^{3a}\psi(q^{24a}))\psi(q^{10a})\psi(q^{2b_1})\cdots\psi(q^{2b_r})$$

and so

$$\sum_{n=0}^{\infty} t(a,15a,8b_1,\ldots,8b_r;4(2n+5a)+a)q^{2n+5a} = 2^{r+3}q^{5a}\psi(q^{40a})\psi(q^{6a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}),$$

$$\sum_{n=0}^{\infty} t(a, 15a, 8b_1, \ldots, 8b_r; 4(2n+3a)+3a)q^{2n+3a} = 2^{r+3}q^{3a}\psi(q^{24a})\psi(q^{10a})\psi(q^{2b_1})\cdots\psi(q^{2b_r}).$$

It then follows that

$$\sum_{n=0}^{\infty} t(a, 15a, 8b_1, \ldots, 8b_r; 4(2n+5a)+a)q^n = 2^{r+3}\psi(q^{20a})\psi(q^{3a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$
$$\sum_{n=0}^{\infty} t(a, 15a, 8b_1, \ldots, 8b_r; 4(2n+3a)+3a)q^n = 2^{r+3}\psi(q^{12a})\psi(q^{5a})\psi(q^{b_1})\cdots\psi(q^{b_r}),$$

which yields (3.16) and (3.17). Finally, (3.18) follows from (2.4) and (3.23), and (3.19) follows from (2.4) and (3.22). The proof is now complete.

# 4. Formulas for $t(a, a, 2a, b; n), t(a, 3a, 3a, 2b; n)$ and $t(a, 6a, 6a, b; n)$

In this section we establish general formulae for $t(a, a, 2a, b; n), t(a, 3a, 3a, 2b; n)$ and $t(a, 6a, 6a, b; n)$, where $a, b, n \in \mathbb{Z}^+$ with $2 \nmid ab$.

**Theorem 4.1.** *Let $a, b \in \mathbb{Z}^+$ with $ab \equiv -1 \pmod 4$. For $n \in \mathbb{Z}^+$,*

$$t(a, a, 2a, b; n) = 2N(a, 4a, 8a, b; 8n+4a+b).$$

Proof. It is clear that

$$\sum_{n=0}^{\infty} N(a, 4a, 8a, b; n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^{4a})\varphi(q^{8a})$$
$$= (\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}) + 2q^a\psi(q^{8a}))(\varphi(q^{16b}) + 2q^{4b}\psi(q^{32b}) + 2q^b\psi(q^{8b}))$$
$$\times (\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}))\varphi(q^{8a}).$$

Thus,

$$\sum_{n=0}^{\infty} N(a, 4a, 8a, b; 8n+4a+b)q^{8n+4a+b}$$
$$= 2q^b\psi(q^{8b})(\varphi(q^{16a}) \cdot 2q^{4a}\psi(q^{32a}) + 2q^{4a}\psi(q^{32a}) \cdot \varphi(q^{16a}))\varphi(q^{8a})$$

and so

$$\sum_{n=0}^{\infty} N(a, 4a, 8a, b; 8n+4a+b)q^n$$
$$= 8\psi(q^b)\varphi(q^{2a})\psi(q^{4a})\varphi(q^a) = 8\psi(q^b)\psi(q^{2a})^2\varphi(q^a) = 8\psi(q^a)^2\psi(q^{2a})\psi(q^b)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} t(a, a, 2a, b; n)q^n.$$

This yields $t(a, a, 2a, b; n) = 2N(a, 4a, 8a, b; 8n+4a+b)$.

**Corollary 4.1.** *Suppose $a, b, n \in \mathbb{Z}^+$ and $ab \equiv -1 \pmod 4$. Then*

$$3N(a, 4a, 8a, b; 8n + 4a + b) = N(a, a, 2a, b; 8n + 4a + b) - N(a, a, 2a, 4b; 8n + 4a + b).$$

Proof. From [11, Theorem 3.1] we know that

$$t(a, a, 2a, b; n) = \frac{2}{3}(N(a, a, 2a, b; 8n + 4a + b) - N(a, a, 2a, 4b; 8n + 4a + b)).$$

This together with Theorem 4.1 yields the result.

**Corollary 4.2.** *Suppose that $n \in \{0, 1, 2, \ldots\}$. Then*

$$N(1, 3, 4, 8; 8n + 7) = \frac{1}{5}N(1, 1, 2, 3; 8n + 7).$$

Proof. By Theorem 4.1, $t(1, 1, 2, 3; n) = 2N(1, 4, 8, 3; 8n + 7)$. By [1], $t(1, 1, 2, 3; n) = \frac{2}{5}N(1, 1, 2, 3; 8n + 7)$. Thus the result follows.

**Theorem 4.2.** *Let $a, b \in \mathbb{Z}^+$ with $2 \nmid ab$. For $n \in \mathbb{Z}^+$ we have*

$$t(a, 3a, 3a, 2b; n) = N(3a, 3a, 4a, 2b; 8n + 7a + 2b).$$

Proof. By (2.5) and (2.6),

$$\sum_{n=0}^{\infty} N(3a, 3a, 4a, 2b; n)q^n = \varphi(q^{3a})^2\varphi(q^{4a})\varphi(q^{2b})$$
$$= (\varphi(q^{12a})^2 + 4q^{6a}\psi(q^{24a})^2 + 4q^{3a}\psi(q^{12a})^2)(\varphi(q^{8b}) + 2q^{2b}\psi(q^{16b}))\varphi(q^{4a}).$$

Thus,

$$\sum_{n=0}^{\infty} N(3a, 3a, 4a, 2b; 4n + 3a + 2b)q^{4n+3a+2b}$$
$$= 4q^{3a}\psi(q^{12a})^2 \cdot 2q^{2b}\psi(q^{16b}) \cdot \varphi(q^{4a}) = 8q^{3a+2b}\varphi(q^{4a})\psi(q^{12a})^2\psi(q^{16b})$$

and so

$$\sum_{n=0}^{\infty} N(3a, 3a, 4a, 2b; 4n + 3a + 2b)q^n = 8\varphi(q^a)\psi(q^{3a})^2\psi(q^{4b}) = 8\varphi(q^a)\varphi(q^{3a})\psi(q^{6a})\psi(q^{4b}).$$

Applying (2.9) we see that

$$\sum_{n=0}^{\infty} N(3a, 3a, 4a, 2b; 4n + 3a + 2b)q^n$$
$$= 8(\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a}) + 2q^a\psi(q^{2a})\psi(q^{6a}))\psi(q^{6a})\psi(q^{4b}).$$

Hence

$$\sum_{n=0}^{\infty} N(3a, 3a, 4a, 2b; 4(2n + a) + 3a + 2b)q^{2n+a} = 16q^a\psi(q^{2a})\psi(q^{6a})^2\psi(q^{4b})$$

and so

$$\sum_{n=0}^{\infty} N(3a,3a,4a,2b;8n+7a+2b)q^n = 16\psi(q^a)\psi(q^{3a})^2\psi(q^{2b}) = \sum_{n=0}^{\infty} t(a,3a,3a,2b;n)q^n,$$

which gives the result.

**Corollary 4.3.** *Suppose $n \in \mathbb{Z}^+$. Then*

$$N(3,3,4,6;8n+13) = 2N(1,3,12,24;8n+13) = \frac{2}{5}N(1,3,3,6;8n+13).$$

Proof. By Theorem 4.2, $t(1,3,3,6;n) = N(3,3,4,6;8n+13)$. By Theorem 4.1, $t(1,3,3,6;n) = 2N(1,3,12,24;8n+13)$. By [10, Theorem 2.10], $t(1,3,3,6;n) = \frac{2}{5}N(1,3,3,6;8n+13)$. Thus, the result follows.

**Corollary 4.4.** *Suppose $a,b,n \in \mathbb{Z}^+$ and $2 \nmid ab$. Then*

$$N(3a,4a,12a,2b;8n+7a+2b) = \frac{1}{2}N(3a,3a,4a,2b;8n+7a+2b), \tag{i}$$

$$\begin{aligned} &N(a,3a,3a,8b;8n+7a+2b) \\ &\quad = N(a,3a,3a,2b;8n+7a+2b) - 2N(3a,3a,4a,2b;8n+7a+2b), \end{aligned} \tag{ii}$$

$$\begin{aligned} &N(a,3a,12a,2b;8n+7a+2b) \\ &\quad = N(a,3a,3a,2b;8n+7a+2b) - \frac{3}{2}N(3a,3a,4a,2b;8n+7a+2b). \end{aligned} \tag{iii}$$

Proof. By [11, Corollary 4.2], $t(a,3a,3a,2b;n) = 2N(4a,12a,3a,2b;8n+7a+2b)$. This together with Theorem 4.2 proves (i). By [11,Theorem 3.1] and Theorem 4.2,

$$\begin{aligned} &N(a,3a,3a,2b;8n+7a+2b) - N(a,3a,3a,8b;8n+7a+2b) \\ &= 2t(a,3a,3a,2b;n) = 2N(3a,3a,4a,2b;8n+7a+2b). \end{aligned}$$

This yields (ii). By [11, Theorem 4.5] and Theorem 4.2,

$$\begin{aligned} &\frac{2}{3}(N(a,3a,3a,2b;8n+7a+2b) - N(a,3a,12a,2b;8n+7a+2b)) \\ &= t(a,3a,3a,2b;n) = N(3a,3a,4a,2b;8n+7a+2b), \end{aligned}$$

which yields part(iii). Hence the corollary is proved.

**Theorem 4.3.** *Let $a,b \in \mathbb{Z}^+$ with $ab \equiv 1 \pmod 4$. For $n \in \mathbb{Z}^+$ we have*

$$t(a,6a,6a,b;n) = \frac{1}{3}N(a,3a,48a,4b;32n+52a+4b).$$

Proof. By (2.9),

$$\begin{aligned} &\sum_{n=0}^{\infty} N(a,3a,48a,4b;n)q^n = \varphi(q^a)\varphi(q^{3a})\varphi(q^{48a})\varphi(q^{4b}) \\ &= (\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a}) + 2q^a(\varphi(q^{12a})\psi(q^{8a}) + q^{2a}\varphi(q^{4a})\psi(q^{24a}))) \end{aligned}$$

$$\times \varphi(q^{48a})\varphi(q^{4b}).$$

Thus,

$$\sum_{n=0}^{\infty} N(a,3a,48a,4b;4n)q^{4n} = (\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a}))\varphi(q^{48a})\varphi(q^{4b}).$$

Appealing to (2.5), (2.7) and (2.9) we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a,3a,48a,4b;4n)q^{n} = (\varphi(q^{a})\varphi(q^{3a}) + 4q^{a}\psi(q^{2a})\psi(q^{6a}))\varphi(q^{12a})\varphi(q^{b}) \\
&= (\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a}) + 6q^{a}(\varphi(q^{12a})\psi(q^{8a}) + q^{2a}\varphi(q^{4a})\psi(q^{24a}))) \\
&\quad \times \varphi(q^{12a})(\varphi(q^{4b}) + 2q^{b}\psi(q^{8b})).
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} N(a,3a,48a,4b;4(4n+a+b))q^{4n+a+b} = 6q^{a}\varphi(q^{12a})\psi(q^{8a})\cdot\varphi(q^{12a})\cdot 2q^{b}\psi(q^{8b})$$

and so

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a,3a,48a,4b;4(4n+a+b))q^{n} \\
&= 12\varphi(q^{3a})^2\psi(q^{2a})\psi(q^{2b}) = 12(\varphi(q^{6a})^2 + 4q^{3a}\psi(q^{12a})^2)\psi(q^{2a})\psi(q^{2b}).
\end{aligned}$$

It then follows that

$$\sum_{n=0}^{\infty} N(a,3a,48a,4b;4(4(2n+3a)+a+b))q^{2n+3a} = 48q^{3a}\psi(q^{12a})^2\psi(q^{2a})\psi(q^{2b})$$

and therefore

$$\sum_{n=0}^{\infty} N(a,3a,48a,4b;4(4(2n+3a)+a+b))q^{n} = 48\psi(q^{a})\psi(q^{6a})^2\psi(q^{b}) = 3\sum_{n=0}^{\infty} t(a,6a,6a,b;n)q^{n}.$$

This yields the result.

# 5. Some special relations between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$

In this section we present many special relations between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$, which were first discovered by calculations with Maple.

**Theorem 5.1.** *For $n \in \mathbb{Z}^+$ we have*

$$\begin{aligned}
&t(2,3,3,4;n) = 2N(2,3,3,4;2n+3) \quad \text{for} \quad n \equiv 2,3 \pmod 4, \\
&t(2,3,3,12;n) = 2N(2,3,3,12;2n+5) \quad \text{for} \quad n \equiv 0,1 \pmod 4, \\
&t(2,3,3,24;n) = 4N(2,3,3,24;2n+8) \quad \text{for} \quad n \equiv 2 \pmod 4,
\end{aligned}$$

$$
\begin{aligned}
t(2,3,3,36;n) &= 2N(2,3,3,36;2n+11) \quad \textit{for} \quad n \equiv 2,3 \pmod 4,\\
t(1,1,6,12;n) &= 2N(1,1,6,12;2n+5) \quad \textit{for} \quad n \equiv 0,3 \pmod 4,\\
t(1,1,6,16;n) &= \begin{cases} N(1,1,3,8;n+3) & \textit{if} \quad n \equiv 2 \pmod 8,\\ 4N(1,1,3,8;n+3) & \textit{if} \quad n \equiv 4 \pmod 8.\end{cases}
\end{aligned}
$$

Proof. By Theorem 4.2 and [12, (3.7)],

$$N(2,3,3,4;8n+9) = t(1,2,3,3;n) = \frac{1}{2}t(2,3,3,4;4n+3).$$

By (2.5) and (2.8),

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2,3,3,4;n)q^n = \varphi(q^2)\varphi(q^4)\varphi(q^3)^2\\
&= (\varphi(q^8)+2q^2\psi(q^{16}))(\varphi(q^{16})+2q^4\psi(q^{32}))\\
&\quad \times (\varphi(q^{24})^2+4q^{12}\psi(q^{48})^2+4q^6\psi(q^{24})^2+4q^3\varphi(q^{48})\psi(q^{24})+8q^{15}\psi(q^{24})\psi(q^{96})).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2,3,3,4;8n+7)q^{8n+7}\\
&= \varphi(q^8)\varphi(q^{16})\cdot 8q^{15}\psi(q^{24})\psi(q^{96}) + \varphi(q^8)\cdot 2q^4\psi(q^{32})\cdot 4q^3\varphi(q^{48})\psi(q^{24})\\
&= 8q^7\varphi(q^8)\psi(q^{24})(\varphi(q^{48})\psi(q^{32})+q^8\varphi(q^{16})\psi(q^{96}))\\
&= 8q^7\varphi(q^8)\psi(q^{24})\psi(q^8)\psi(q^{24})
\end{aligned}
$$

and so

$$\sum_{n=0}^{\infty} N(2,3,3,4;8n+7)q^n = 8\varphi(q)\psi(q)\psi(q^3)^2. \tag{5.1}$$

On the other hand, using (2.4), (2.5) and (2.7) we see that for $b \in \mathbb{Z}^+$,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} t(2,3,3,4b;n)q^n\\
&= 16\psi(q^2)\psi(q^3)^2\psi(q^{4b}) = 16\varphi(q^3)\psi(q^2)\psi(q^6)\psi(q^{4b})\\
&= 16(\varphi(q^{12})+2q^3\psi(q^{24}))(\varphi(q^{12})\psi(q^8)+q^2\varphi(q^4)\psi(q^{24}))\psi(q^{4b}).
\end{aligned} \tag{5.2}
$$

Thus

$$\sum_{n=0}^{\infty} t(2,3,3,4b;4n+2)q^{4n+2} = 16q^2\varphi(q^4)\varphi(q^{12})\psi(q^{24})\psi(q^{4b})$$

and so

$$\sum_{n=0}^{\infty} t(2,3,3,4b;4n+2)q^n = 16\varphi(q)\varphi(q^3)\psi(q^6)\psi(q^b) = 16\varphi(q)\psi(q^3)^2\psi(q^b).$$

This together with (5.1) yields $t(2,3,3,4;4n+2) = 2N(2,3,3,4;8n+7)$. Therefore $t(2,3,3,4;n) = 2N(2,3,3,4;2n+3)$ for $n \equiv 2,3 \pmod 4$. The remaining results can be proved similarly.

**Lemma 5.1.** *Suppose $a, b \in \mathbb{Z}^+$ and $ab \equiv -1 \pmod 4$. Then*

$$\sum_{n=0}^{\infty} N(a,a,a,2b;8n+5a)q^n = 24\varphi(q^b)\psi(q^a)\psi(q^{2a})^2,$$
$$\sum_{n=0}^{\infty} N(a,a,a,2b;8n+a+2b)q^n = 12\varphi(q^a)^2\psi(q^a)\psi(q^{2b}).$$

Proof. By (2.5),

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a,a,a,2b;n)q^n\\ &= \varphi(q^a)^3\varphi(q^{2b}) = (\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}) + 2q^a\psi(q^{8a}))^3(\varphi(q^{8b}) + 2q^{2b}\psi(q^{16b}))\\ &= \big((\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}))^3 + 6q^a(\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}))^2\psi(q^{8a}) + 12q^{2a}(\varphi(q^{16a})\\ &\quad + 2q^{4a}\psi(q^{32a}))\psi(q^{8a})^2 + 8q^{3a}\psi(q^{8a})^3\big)(\varphi(q^{8b}) + 2q^{2b}\psi(q^{16b})).\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a,a,a,2b;8n+5a)q^{8n+5a}\\ &\quad = 6q^a\cdot 4q^{4a}\varphi(q^{16a})\psi(q^{32a})\psi(q^{8a})\varphi(q^{8b}) = 24q^{5a}\psi(q^{16a})^2\psi(q^{8a})\varphi(q^{8b}),\\ &\sum_{n=0}^{\infty} N(a,a,a,2b;8n+a+2b)q^{8n+a+2b}\\ &\quad = 6q^a(\varphi(q^{16a})^2 + 4q^{8a}\psi(q^{32a})^2)\psi(q^{8a})\cdot 2q^{2b}\psi(q^{16b}) = 12q^{a+2b}\varphi(q^{8a})^2\psi(q^{8a})\psi(q^{16b})\end{aligned}$$

and so the result follows.

**Theorem 5.2.** *For $n \in \mathbb{Z}^+$ with $n \equiv 3, 5 \pmod 8$,*

$$t(1,1,2,12;n) = 4N(1,1,4,6;n+2) = \frac{8}{3}N(1,1,1,6;n+2),$$
$$t(3,3,4,6;n) = \frac{8}{3}N(2,3,3,3;n+2).$$

Proof. By (1.3) and [10, Theorem 2.11],

$$t(1,1,2,12;8n+3) = 2t(1,1,4,6;4n+1) = 4N(1,1,4,6;8n+5),$$
$$t(1,1,2,12;8n+5) = 2t(1,1,4,6;4n+2) = 4N(1,1,4,6;8n+7).$$

From [10, p.283] and Lemma 5.1 (with $a = 1$ and $b = 3$) we know that

$$\sum_{n=0}^{\infty} t(1,1,4,6;4n+1)q^n = 32\varphi(q^3)\psi(q)\psi(q^2)^2 = \frac{4}{3}\sum_{n=0}^{\infty} N(1,1,1,6;8n+5)q^n,$$
$$\sum_{n=0}^{\infty} t(1,1,4,6;4n+2)q^n = 16\varphi(q)^2\psi(q)\psi(q^6) = \frac{4}{3}\sum_{n=0}^{\infty} N(1,1,1,6;8n+7)q^n,$$

which yields

$$t(1,1,4,6;4n+1)=\frac{4}{3}N(1,1,1,6;8n+5),\quad t(1,1,4,6;4n+2)=\frac{4}{3}N(1,1,1,6;8n+7).$$

Hence the formula for $t(1,1,2,12;n)$ is true. From (5.2) we see that

$$\sum_{n=0}^{\infty}t(2,3,3,12;4n)q^{4n}=16\varphi(q^{12})^2\psi(q^8)\psi(q^{12}),$$
$$\sum_{n=0}^{\infty}t(2,3,3,12;4n+5)q^{4n+5}=32q^5\varphi(q^4)\psi(q^{24})^2\psi(q^{12}).$$

Thus, appealing to Lemma 5.1 (with $a=3$ and $b=1$) we get

$$\sum_{n=0}^{\infty}t(2,3,3,12;4n)q^{n}=16\varphi(q^3)^2\psi(q^2)\psi(q^3)=\frac{4}{3}\sum_{n=0}^{\infty}N(2,3,3,3;8n+5)q^n,$$
$$\sum_{n=0}^{\infty}t(2,3,3,12;4n+5)q^{n}=32\varphi(q)\psi(q^3)\psi(q^6)^2=\frac{4}{3}\sum_{n=0}^{\infty}N(2,3,3,3;8n+15)q^n.$$

This together with (1.3) gives

$$t(3,3,4,6;8n+3)=2t(2,3,3,12;4n)=\frac{8}{3}N(2,3,3,3;8n+5),$$
$$t(3,3,4,6;8n+5)=2t(2,3,3,12;4n+1)=\frac{8}{3}N(2,3,3,3;8n+7),$$

which completes the proof.

**Corollary 5.1.** *For $n\in\mathbb{Z}^+$ with $n\equiv 5,7 \pmod 8$,*

$$N(1,1,1,6;4n)=5N(1,1,1,6;n)\quad \textit{and}\quad N(2,3,3,3;4n)=5N(2,3,3,3;n).$$

Proof. From Theorem 5.2 and [10, Theorem 2.1],

$$\frac{8}{3}N(1,1,1,6;n)=t(1,1,2,12;n-2)=\frac{2}{3}(N(1,1,1,6;4n)-N(1,1,1,6;n)),$$
$$\frac{8}{3}N(2,3,3,3;n)=t(3,3,4,6;n-2)=\frac{2}{3}(N(2,3,3,3;4n)-N(2,3,3,3;n)).$$

This yields the result.

Using similar method one can prove the following results:

$$t(3,3,4,18;n)=2N(3,3,4,18;2n+7)\quad\text{for}\quad n\equiv 0,1 \pmod 4, \tag{5.3}$$
$$t(1,3,8,12;n)=4N(1,3,8,12;n+3)\quad\text{for}\quad n\equiv 2,4 \pmod 8, \tag{5.4}$$
$$t(1,1,2,28;n)=4N(1,1,2,28;n+4)\quad\text{for}\quad n\equiv 1,3 \pmod 8, \tag{5.5}$$
$$t(1,3,4,24;n)=4N(1,3,4,24;n+4)\quad\text{for}\quad n\equiv 1,3 \pmod 8, \tag{5.6}$$
$$t(2,3,3,48;n)=N(2,3,3,48;2n+14)\quad\text{for}\quad n\equiv 0 \pmod 8, \tag{5.7}$$
$$t(1,1,8,14;n)=8N(1,1,8,14;n+3)\quad\text{for}\quad n\equiv 1 \pmod 8, \tag{5.8}$$

$$t(1,1,10,20;n) = 4N(1,1,10,20;n+4) \quad \text{for} \quad n \equiv 1 \pmod 8, \tag{5.9}$$
$$t(1,1,14,16;n) = 4N(1,1,14,16;n+4) \quad \text{for} \quad n \equiv 1 \pmod 8, \tag{5.10}$$
$$t(1,2,7,14;n) = 8N(1,2,7,14;n+3) \quad \text{for} \quad n \equiv 1 \pmod 8, \tag{5.11}$$
$$t(1,1,8,30;n) = 4N(1,1,8,30;2n+10) \quad \text{for} \quad n \equiv 1 \pmod 8, \tag{5.12}$$
$$t(1,3,4,16;n) = \frac{4}{3}N(1,3,4,16;2n+6) \quad \text{for} \quad n \equiv 1 \pmod 8, \tag{5.13}$$
$$t(3,3,10,48;n) = 4N(3,3,10,48;2n+16) \quad \text{for} \quad n \equiv 1 \pmod 8, \tag{5.14}$$
$$t(1,1,8,14;n) = 4N(1,1,8,14;2n+6) \quad \text{for} \quad n \equiv 3 \pmod 8, \tag{5.15}$$
$$t(2,15,15,24;n) = 4N(2,15,15,24;2n+14) \quad \text{for} \quad n \equiv 3 \pmod 8, \tag{5.16}$$
$$t(5,5,6,8;n) = 4N(5,5,6,8;2n+6) \quad \text{for} \quad n \equiv 3 \pmod 8, \tag{5.17}$$
$$t(1,3,12,48;n) = \frac{4}{3}N(1,3,12,48;2n+16) \quad \text{for} \quad n \equiv 4 \pmod 8, \tag{5.18}$$
$$t(2,4,5,5;n) = 4N(2,4,5,5;n+2) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.19}$$
$$t(4,7,7,14;n) = 4N(4,7,7,14;n+4) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.20}$$
$$t(1,1,16,30;n) = 4N(1,1,16,30;2n+12) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.21}$$
$$t(1,1,30,40;n) = 4N(1,1,30,40;2n+18) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.22}$$
$$t(1,3,16,36;n) = \frac{4}{3}N(1,3,16,36;2n+14) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.23}$$
$$t(2,3,3,32;n) = 4N(2,3,3,32;2n+10) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.24}$$
$$t(2,7,7,24;n) = 4N(2,7,7,24;2n+10) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.25}$$
$$t(3,3,10,24;n) = 4N(3,3,10,24;2n+10) \quad \text{for} \quad n \equiv 5 \pmod 8, \tag{5.26}$$
$$t(1,7,16,16;n) = 4N(1,7,16,16;n+5) \quad \text{for} \quad n \equiv 6 \pmod 8, \tag{5.27}$$
$$t(2,3,3,48;n) = 4N(2,3,3,48;2n+14) \quad \text{for} \quad n \equiv 6 \pmod 8, \tag{5.28}$$
$$t(1,1,10,20;n) = 4N(1,1,10,20;n+4) \quad \text{for} \quad n \equiv 7 \pmod 8, \tag{5.29}$$
$$t(2,4,5,5;n) = 4N(2,4,5,5;n+2) \quad \text{for} \quad n \equiv 7 \pmod 8, \tag{5.30}$$
$$t(4,7,7,14;n) = 4N(4,7,7,14;n+4) \quad \text{for} \quad n \equiv 7 \pmod 8, \tag{5.31}$$
$$t(1,1,14,16;n) = 4N(1,1,14,16;2n+8) \quad \text{for} \quad n \equiv 7 \pmod 8, \tag{5.32}$$
$$t(5,5,6,40;n) = 4N(5,5,6,40;2n+14) \quad \text{for} \quad n \equiv 7 \pmod 8. \tag{5.33}$$

By doing calculations with Maple, we pose the following conjectures.
**Conjecture 5.1.** *Let $n \in \mathbb{Z}^+$. Then*

$$t(1,2,3,10;n) = \begin{cases} \frac{4}{3}N(1,2,3,10;2n+4) = \frac{4}{9}N(1,2,3,10;8n+16) & \text{if } 4 \mid n-1, \\ \frac{8}{3}N(1,2,3,10;2n+4) & \text{if } 8 \mid n, \\ \frac{16}{9}N(1,2,3,10;2n+4) & \text{if } n \equiv 10 \pmod{16}, \\ 4N(1,2,3,10;2n+4) & \text{if } n \equiv 11,15 \pmod{20}. \end{cases}$$

**Conjecture 5.2.** *Let* $n \in \mathbb{Z}^+$. *Then*

$$t(1,2,3,18;n) = \begin{cases} \frac{4}{3}N(1,2,3,18;2n+6) = \frac{4}{9}N(1,2,3,18;8n+24) & \text{if } 4 \mid n, \\ \frac{8}{3}N(1,2,3,18;2n+6) & \text{if } 8 \mid n-3, \\ 4N(1,2,3,18;2n+6) & \text{if } 12 \mid n-6, \\ \frac{8}{5}N(1,2,3,18;2n+6) & \text{if } 24 \mid n-15, \end{cases}$$

**Conjecture 5.3.** *Let* $n \in \mathbb{Z}^+$. *Then*

$$t(1,3,6,30;n) = \begin{cases} \frac{4}{3}N(1,3,6,30;2n+10) = \frac{4}{9}N(1,3,6,30;8n+40) & \text{if } 4 \mid n, \\ \frac{8}{3}N(1,3,6,30;2n+10) = \frac{8}{15}N(1,3,6,30;8n+40) & \text{if } 8 \mid n-1, \\ \frac{16}{9}N(1,3,6,30;2n+10) = \frac{16}{33}N(1,3,6,30;8n+40) & \text{if } 16 \mid n+1. \end{cases}$$

**Conjecture 5.4.** *Let* $n \in \mathbb{Z}^+$. *Then*

$$t(1,3,18,18;n) = \begin{cases} \frac{4}{3}N(1,3,18,18;2n+10) = \frac{4}{9}N(1,3,18,18;8n+40) & \text{if } 4 \mid n-2, \\ \frac{8}{3}N(1,3,18,18;2n+10) = \frac{8}{15}N(1,3,18,18;8n+40) & \text{if } 8 \mid n-1, \\ \frac{16}{9}N(1,3,18,18;2n+10) = \frac{16}{33}N(1,3,18,18;8n+40) & \text{if } 16 \mid n-7, \\ 4N(1,3,18,18;2n+10) = \frac{4}{7}N(1,3,18,18;8n+40) & \text{if } 12 \mid n-4, \\ \frac{8}{5}N(1,3,18,18;2n+10) = \frac{8}{17}N(1,3,18,18;8n+40) & \text{if } 24 \mid n-13. \end{cases}$$

**Conjecture 5.5.** *Let* $n \in \mathbb{Z}^+$. *Then*

$$t(2,3,9,18;n) = \begin{cases} \frac{4}{3}N(2,3,9,18;2n+8) = \frac{4}{9}N(2,3,9,18;8n+32) & \text{if } 4 \mid n-3, \\ \frac{8}{3}N(2,3,9,18;2n+8) & \text{if } 8 \mid n-2, \\ \frac{16}{9}N(2,3,9,18;2n+8) & \text{if } 16 \mid n-8, \\ \frac{32}{15}N(2,3,9,18;2n+8) & \text{if } 32 \mid n-20, \\ \frac{8}{5}N(2,3,9,18;2n+8) & \text{if } 24 \mid n-14, \\ 4N(2,3,9,18;2n+8) & \text{if } 12 \mid n-5. \end{cases}$$

**Conjecture 5.6.** *Let* $n \in \mathbb{Z}^+$. *Then*

$t(2,5,10,15;n)$

$$= \begin{cases} \frac{4}{3}N(2,5,10,15;2n+8) = \frac{4}{9}N(2,5,10,15;8n+32) & \textit{if } 4 \mid n-3, \\ \frac{8}{3}N(2,5,10,15;2n+8) & \textit{if } 8 \mid n-6, \\ \frac{16}{9}N(2,5,10,15;2n+8) & \textit{if } 16 \mid n-8, \\ 4N(2,5,10,15;2n+8) & \textit{if } n \equiv 61, 81 \pmod{100}. \end{cases}$$

**Conjecture 5.7.** *Let $n \in \mathbb{Z}^+$. Then*

$$t(5,6,15,30;n) = \begin{cases} \frac{4}{3}N(5,6,15,30;2n+14) & \textit{if } 4 \mid n-2, \\ \frac{8}{3}N(5,6,15,30;2n+14) & \textit{if } 8 \mid n-7, \\ \frac{16}{9}N(5,6,15,30;2n+14) & \textit{if } 16 \mid n-13. \end{cases}$$

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