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Congruences for Catalan-Larcombe-French numbers

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Abstract

Let $\{P_n\}$ be the Catalan-Larcombe-French numbers given by $P_0 = 1,\ P_1 = 8$ and $n^2P_n = 8(3n^2-3n+1)P_{n-1}-128(n-1)^2P_{n-2}$ $(n \ge 2)$, and let $S_n = P_n/2^n$. In this paper we deduce congruences for $S_{np}, S_{np+1} \pmod{p^3}$, $S_{mp^r-1} \pmod{p^r}$ and $S_{mp^r+1} \pmod{p^{2r}}$, where p is an odd prime and m, n, r are positive integers.

MSC: Primary 11A07, Secondary 05A10, 05A19, 11B68, 11B83
Keywords: congruence; Catalan-Larcombe-French number

1. Introduction

Let $\{P_n\}$ be the sequence given by

(1.1) $\quad P_0 = 1,\ P_1 = 8 \quad \text{and} \quad (n+1)^2P_{n+1} = 8(3n^2+3n+1)P_n - 128n^2P_{n-1}\ (n \ge 1).$

The numbers P_n are called Catalan-Larcombe-French numbers since Catalan first defined P_n in [C], and in [LF1] Larcombe and French proved that

$$P_n = 2^n\sum_{k=0}^{[n/2]}(-4)^k\binom{2n-2k}{n-k}^2\binom{n-k}{k} = \sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{2n-2k}{n-k}^2}{\binom{n}{k}}, \tag{1.2}$$

where $[x]$ is the greatest integer not exceeding x. The numbers P_n occur in the theory of elliptic integrals, and are related to the arithmetic-geometric-mean. See [LF1] and A053175 in Sloane's database "The On-Line Encyclopedia of Integer Sequences".

Let $\{S_n\}$ be defined by

(1.3) $\quad S_0 = 1,\ S_1 = 4 \quad \text{and} \quad (n+1)^2S_{n+1} = 4(3n^2+3n+1)S_n - 32n^2S_{n-1}\ (n \ge 1).$

Comparing (1.3) with (1.1), we see that

$$S_n = \frac{P_n}{2^n}. \tag{1.4}$$

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In 2009, Zagier[Z] noted that

$$S_n=\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{n}{2k}4^{n-2k}. \tag{1.5}$$

In this paper we investigate the properties of S_n instead of P_n since S_n is an Apéry-like sequence. As observed by V. Jovovic in 2003 (see [LF2]),

$$S_n=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\quad (n=0,1,2,\ldots). \tag{1.6}$$

Recently Z.W. Sun stated that (see A053175 in Sloane's database OEIS)

$$S_n=\frac{1}{(-2)^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{k}{n-k}(-4)^k. \tag{1.7}$$

The first few values of S_n are shown below:

$$S_0=1,\ S_1=4,\ S_2=20,\ S_3=112,\ S_4=676,\ S_5=4304,$$
$$S_6=28496,\ S_7=194240,\ S_8=1353508,\ S_9=9593104.$$

Let p be an odd prime. In [JLF], Jarvis, Larcombe and French proved that if $n=a_rp^r+\cdots+a_1p+a_0$ with $a_0,a_1,\ldots,a_r\in\{0,1,\ldots,p-1\}$, then

$$P_n\equiv P_{a_r}\cdots P_{a_1}P_{a_0}\pmod p. \tag{1.8}$$

In [JV] Jarvis and Verrill showed that

$$P_n\equiv(-1)^{\frac{p-1}{2}}128^nP_{p-1-n}\pmod p\quad\text{for}\quad n=0,1,\ldots,p-1 \tag{1.9}$$

and

$$P_{mp^r}\equiv P_{mp^{r-1}}\pmod{p^r}\quad\text{for}\quad m,r\in\mathbb{Z}^+, \tag{1.10}$$

where $\mathbb{Z}^+$ is the set of positive integers. In [OS] Osburn and Sahu stated that

$$S_{mp^r}\equiv S_{mp^{r-1}}\pmod{p^{2r}}\quad\text{for}\quad m,r\in\mathbb{Z}^+. \tag{1.11}$$

For a prime p let $\mathbb{Z}_p$ denote the set of those rational numbers whose denominator is not divisible by p. Let p be an odd prime, $n\in\mathbb{Z}_p$ and $n\not\equiv 0,-16\pmod p$. In [S2] the second author proved that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(n+16)^k}\equiv\Big(\frac{n(n+16)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}}\pmod p, \tag{1.12}$$

where $(\frac{a}{p})$ is the Legendre symbol.

Let $\{E_n\}$ be the Euler numbers given by

$$E_{2n-1}=0,\ E_0=1\quad\text{and}\quad\sum_{k=0}^{n}\binom{2n}{2k}E_{2k}=0\ (n\ge 1).$$

Suppose that $p>3$ is a prime, $n\in\mathbb{Z}^+$ and $p\nmid n$. In this paper we show that

$$S_{np}\equiv S_n+8n^2S_{n-1}(-1)^{\frac{p-1}{2}}p^2E_{p-3}\pmod{p^3}. \tag{1.13}$$

We also determine $S_{np+1}\pmod{p^3}$ and show that for $m,r\in\mathbb{Z}^+$,
(1.14)

$$S_{mp^r+1}\equiv 4(mp^r+1)S_{mp^{r-1}}\pmod{p^{2r}}\ \text{and}\ S_{mp^r-1}\equiv(-1)^{\frac{p-1}{2}}S_{mp^{r-1}-1}\pmod{p^r}.$$

Throughout this paper, $\mathrm{ord}_p n$ is the unique nonnegative integer α such that $p^\alpha\mid n$ and $p^{\alpha+1}\nmid n$.

2. Basic lemmas

Lemma 2.1 (Lucas theorem [M]). *Let p be an odd prime. Suppose $a=a_rp^r+\cdots+a_1p+a_0$ and $b=b_rp^r+\cdots+b_1p+b_0$, where $a_r,\dots,a_0,b_r,\dots,b_0\in\{0,1,\dots,p-1\}$. Then*

$$\binom{a}{b}\equiv\binom{a_r}{b_r}\cdots\binom{a_0}{b_0}\pmod p.$$

Lucas theorem is often formulated as follows.

Lemma 2.2 ([M]). *Let p be an odd prime and $a,b\in\mathbb{Z}^+$. Suppose $a_0,b_0\in\{0,1,\dots,p-1\}$. Then*

$$\binom{ap+a_0}{bp+b_0}\equiv\binom{a}{b}\binom{a_0}{b_0}\pmod p.$$

Lemma 2.3 (Ljunggren's congruence [M, (22)]). *Let $p\ge 5$ be a prime and $m,n\in\mathbb{Z}^+$. Then*

$$\binom{mp}{np}\equiv\binom{m}{n}\pmod{p^3}.$$

Lemma 2.4 ([Su, Lemma 2.1]). *Let p be an odd prime and $k\in\{1,2,\dots,p-1\}$. Then*

$$\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv\begin{cases}-\dfrac{2p}{k}\pmod{p^2} & \text{if } k<\dfrac p2,\\ \dfrac{2p}{k}\pmod{p^2} & \text{if } k>\dfrac p2.\end{cases}$$

Let $\{B_n\}$ be the Bernoulli numbers defined by $B_0=1$ and $\sum_{k=0}^{n-1}\binom nk B_k=0\ (n\ge 2)$. It is known that $B_{2k+1}=0$ for $k\in\mathbb{Z}^+$. For $m,n\in\mathbb{Z}^+$ it is well known that

$$\sum_{k=0}^{n-1}k^m=\frac{1}{m+1}\sum_{k=1}^{m+1}\binom{m+1}{k}B_{m+1-k}n^k. \tag{2.1}$$

By the Staudt-Clausen theorem, $B_{2k}\in\mathbb{Z}_p$ for $2k\not\equiv 0\pmod{p-1}$, and $pB_{2k}\in\mathbb{Z}_p$ for $2k\equiv 0\pmod{p-1}$. See [MOS].

Let $\{E_n(x)\}$ be the Euler polynomials given by

$$E_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom nk(2x-1)^{n-k}E_k.$$

Then $E_n = 2^n E_n(\frac{1}{2})$. It is known that (see [MOS])

$$\sum_{k=0}^{n-1}(-1)^k k^m = \frac{E_m(0) - (-1)^n E_m(n)}{2}.$$

Lemma 2.5 ([S1, Lemma 2.2]). *Let p be an odd prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod p$ and $k \in \{1, 2, \ldots, p-2\}$. Then*

$$\sum_{r=1}^{\langle a\rangle_p} \frac{(-1)^r}{r^k} \equiv -\frac{(2^{p-k}-1)B_{p-k}}{p-k} + \frac{1}{2}(-1)^{\langle a\rangle_p + k} E_{p-1-k}(-a) \pmod p,$$

where $\langle a\rangle_p \in \{0, 1, \ldots, p-1\}$ is given by $a \equiv \langle a\rangle_p \pmod p$.

Lemma 2.6. *Let p be an odd prime, $k, m \in \mathbb{Z}^+$ and $\frac{p}{2} < k < p$. Then*

$$\binom{2mp+2k}{mp+k} \equiv (2m+1)\binom{2m}{m}\binom{2k}{k} \pmod{p^2}.$$

Proof. Clearly

$$\begin{aligned}
&\frac{(2m+1)p((2m+1)p-1)\cdots((m+1)p+1)(m+1)p}{(mp)!} \\
&= \frac{((2m+1)p)(2mp)\cdots((m+1)p)}{p\cdot(2p)\cdots(mp)} \cdot \frac{\prod_{r=m+1}^{2m}(rp+1)\cdots(rp+p-1)}{\prod_{r=0}^{m-1}(rp+1)\cdots(rp+p-1)} \\
&= p(2m+1)\binom{2m}{m} \cdot \frac{\prod_{r=m+1}^{2m}(rp+1)\cdots(rp+p-1)}{\prod_{r=0}^{m-1}(rp+1)\cdots(rp+p-1)} \\
&\equiv p(2m+1)\binom{2m}{m}\frac{(p-1)!^m}{(p-1)!^m} = p(2m+1)\binom{2m}{m} \pmod{p^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\binom{2mp+2k}{mp+k} \\
&= \frac{(2m+1)p((2m+1)p-1)\cdots((m+1)p+1)(m+1)p}{(mp)!} \\
&\quad \times \frac{(2mp+2k)\cdots(2mp+p+1)((m+1)p-1)\cdots((m+1)p-(p-1-k))}{(mp+1)\cdots(mp+k)} \\
&\equiv p(2m+1)\binom{2m}{m}\frac{(2k)(2k-1)\cdots(p+1)(p-1)(p-2)\cdots(k+1)}{k!} \\
&= (2m+1)\binom{2m}{m}\binom{2k}{k} \pmod{p^2}.
\end{aligned}$$

This proves the lemma.

Lemma 2.7. *For any positive integer n we have*

$$S_n = 2\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{2k}{k}\binom{2n-2k}{n-k}.$$

Proof. Since we have, replacing k by $n-k$ for the first equality,

$$\begin{aligned}\sum_{k=0}^{n}(2k-n)\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} &= \sum_{k=0}^{n}(2(n-k)-n)\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\\ &= -\sum_{k=0}^{n}(2k-n)\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k},\end{aligned}$$

we see that

$$\sum_{k=0}^{n}(2k-n)\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}=0 \tag{2.2}$$

and so from (1.6)

$$nS_n=\sum_{k=0}^{n}2k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}=2n\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{2k}{k}\binom{2n-2k}{n-k}.$$

This yields the result.

Lemma 2.8. *Let $m\in\mathbb{Z}$ and $k,n,p\in\mathbb{Z}^+$. Then*

$$\binom{mp^r-1}{k}=(-1)^{k-[\frac{k}{p}]}\binom{mp^{r-1}-1}{[k/p]}\prod_{\substack{i=1\\ p\nmid i}}^{k}\Big(1-\frac{mp^r}{i}\Big)$$

and

$$\binom{mp^r}{np}=(-1)^{n(p-1)}\binom{mp^{r-1}}{n}\prod_{\substack{i=1\\ p\nmid i}}^{np-1}\Big(1-\frac{mp^r}{i}\Big).$$

Proof. Clearly,

$$\begin{aligned}\binom{mp^r-1}{k}&=\prod_{i=1}^{k}\frac{mp^r-i}{i}=\prod_{\substack{i=1\\ p\nmid i}}^{k}\frac{mp^r-i}{i}\prod_{i=1}^{[k/p]}\frac{mp^r-pi}{pi}\\ &=\prod_{\substack{i=1\\ p\nmid i}}^{k}\frac{mp^r-i}{i}\prod_{i=1}^{[k/p]}\frac{mp^{r-1}-i}{i}\\ &=(-1)^{k-[\frac{k}{p}]}\prod_{\substack{i=1\\ p\nmid i}}^{k}\Big(1-\frac{mp^r}{i}\Big)\cdot\binom{mp^{r-1}-1}{[k/p]},\end{aligned}$$

establishing the first identity. Taking $k=np-1$ in the above we see that

$$\binom{mp^r}{np}=\frac{mp^r}{np}\binom{mp^r-1}{np-1}=\frac{mp^{r-1}}{n}\binom{mp^{r-1}-1}{n-1}\cdot(-1)^{np-1-(n-1)}\prod_{\substack{i=1\\ p\nmid i}}^{np-1}\Big(1-\frac{mp^r}{i}\Big).$$

This yields the second identity.

Lemma 2.9 ([SD, proof of Lemma 3.2]). *Let $m,n\in\mathbb{Z}^+$. Then*

$$\binom{3m}{3n}\equiv\binom{m}{n}(1+9mn^2-9m^2n)\pmod{27}.$$

Lemma 2.10. *Let p be an odd prime, $r,m\in\mathbb{Z}^+$ and $s\in\{0,1,\ldots,mp^{r-1}\}$. Then*

$$\binom{mp^r}{sp}\equiv\binom{mp^{r-1}}{s}\pmod{p^{2r}}.$$

Proof. Clearly the result is true for $s=0$. Now we assume $s\ge 1$. By Lemma 2.8,

$$\binom{mp^r}{sp}=\binom{mp^{r-1}}{s}\prod_{\substack{i=1\\ p\nmid i}}^{sp-1}\Big(1-\frac{mp^r}{i}\Big)\equiv\binom{mp^{r-1}}{s}\Big(1-mp^r\sum_{\substack{i=1\\ p\nmid i}}^{sp-1}\frac{1}{i}\Big)\pmod{p^{2r}}.$$

Let $\varphi(n)$ be the Euler's totient function. Set $l=\mathrm{ord}_p s$ and $s=p^l s_0$. Since $B_1=-\frac{1}{2}$, $B_{2s+1}=0\ (s\ge 1)$, $pB_k\in\mathbb{Z}_p$ and $\varphi(p^{l+1})\ge l+2$ we see that

$$\begin{aligned}\sum_{\substack{i=1\\ p\nmid i}}^{sp-1}\frac{1}{i}&\equiv\sum_{\substack{i=1\\ p\nmid i}}^{sp-1}i^{\varphi(p^{l+1})-1}\equiv\sum_{i=1}^{sp-1}i^{\varphi(p^{l+1})-1}\\ &=\frac{1}{\varphi(p^{l+1})}\sum_{j=1}^{\varphi(p^{l+1})}\binom{\varphi(p^{l+1})}{j}(s_0p^{l+1})^jB_{\varphi(p^{l+1})-j}\\ &=(s_0p^{l+1})^{\varphi(p^{l+1})-1}B_1+\frac{1}{\varphi(p^{l+1})}\sum_{j=1}^{\varphi(p^{l+1})/2}\binom{\varphi(p^{l+1})}{2j}(s_0p^{l+1})^{2j}B_{\varphi(p^{l+1})-2j}\\ &\equiv\frac{s_0}{p-1}\sum_{j=1}^{\varphi(p^{l+1})/2}\binom{\varphi(p^{l+1})}{2j}(s_0p^{l+1})^{2j-1}\cdot pB_{\varphi(p^{l+1})-2j}\equiv 0\pmod{p^{l+1}}.\end{aligned}\tag{2.3}$$

If $l\ge r-1$, then $r+l+1\ge 2r$ and so

$$\binom{mp^r}{sp}\equiv\binom{mp^{r-1}}{s}\Big(1-mp^r\sum_{\substack{i=1\\ p\nmid i}}^{sp-1}\frac{1}{i}\Big)\equiv\binom{mp^{r-1}}{s}\pmod{p^{2r}}.$$

If $0\le l<r-1$, then $\binom{mp^{r-1}}{s}=\frac{mp^{r-1}}{s_0p^l}\binom{mp^{r-1}-1}{s-1}\equiv 0\pmod{p^{r-1-l}}$ and so

$$\binom{mp^r}{sp}\equiv\binom{mp^{r-1}}{s}-mp^r\binom{mp^{r-1}}{s}\sum_{\substack{i=1\\ p\nmid i}}^{sp-1}\frac{1}{i}\equiv\binom{mp^{r-1}}{s}\pmod{p^{2r}}.$$

This completes the proof.

Lemma 2.11. *Let p be an odd prime, $r,m\in\mathbb{Z}^+$ and $s\in\{0,1,\ldots,mp^{r-1}\}$. Then*

$$\binom{mp^{r-1}}{s}\binom{2sp}{sp}\binom{2(mp^{r-1}-s)p}{(mp^{r-1}-s)p}$$

$$\equiv \begin{cases} \binom{m}{s}\binom{2s}{s}\binom{2(m-s)}{m-s}(1+9m) \pmod{p^{r+2}} & \text{if } r=1 \text{ and } p=3, \\ \binom{mp^{r-1}}{s}\binom{2s}{s}\binom{2(mp^{r-1}-s)}{mp^{r-1}-s} \pmod{p^{r+2}} & \text{if } r>1 \text{ or } p>3. \end{cases}$$

Proof. Clearly the result is true for $s=0$. Now we assume $s\ge 1$. For $r=1$ the result follows from Lemmas 2.3 and 2.9. Now assume $r\ge 2$. If $p\nmid s$, then $\binom{mp^{r-1}}{s} = \frac{mp^{r-1}}{s}\binom{mp^{r-1}-1}{s-1}\equiv 0 \pmod{p^{r-1}}$. By Lemmas 2.3 and 2.9,

$$\binom{2sp}{sp}\binom{2(mp^{r-1}-s)p}{(mp^{r-1}-s)p} \equiv \binom{2s}{s}\binom{2(mp^{r-1}-s)}{mp^{r-1}-s} \pmod{p^3}.$$

Thus the result is true when $p\nmid s$. Now assume that $p\mid s$, $l=\mathrm{ord}_p s$ and $s=p^l s_0$. For $1\le l<r-1$, using Lemma 2.10 we see that

$$\binom{2sp}{sp}\binom{2(mp^{r-1}-s)p}{(mp^{r-1}-s)p} \equiv \binom{2s}{s}\binom{2(mp^{r-1}-s)}{mp^{r-1}-s} \pmod{p^{2l+2}}.$$

Since $\binom{mp^{r-1}}{s} = \frac{mp^{r-1}}{p^l s_0}\binom{mp^{r-1}-1}{s-1}\equiv 0 \pmod{p^{r-1-l}}$ and $r-1-l+2l+2=r+l+1\ge r+2$, the result is true in this case. For $l\ge r-1$ we see that $p^r\mid sp$ and $p^r\mid (mp^{r-1}-s)p$. Thus applying Lemma 2.10 we deduce that

$$\binom{2sp}{sp}\binom{2(mp^{r-1}-s)p}{(mp^{r-1}-s)p} \equiv \binom{2s}{s}\binom{2(mp^{r-1}-s)}{mp^{r-1}-s} \pmod{p^{2r}}.$$

As $2r\ge r+2$, the result is again true. The proof is now complete.

Lemma 2.12. *Let p be an odd prime, $m,r\in\mathbb{Z}^+$ and $k\in\{0,1,\ldots,mp^r\}$. Then*

$$k\binom{2k}{k}\binom{2(mp^r-k)}{mp^r-k}\equiv 0 \pmod{p^r}.$$

Proof. Clearly the result is true for $k=0$. Now we suppose $k\ge 1$. Suppose $s=[\frac{k}{p}]$ and $t=k-sp$. Then $t\in\{0,1,\ldots,p-1\}$. We first assume $p\nmid k$. That is, $t>0$. Let us consider the case $r=1$. By Lemma 2.2, for $1\le t<\frac{p}{2}$,

$$\begin{aligned}\binom{2k}{k}\binom{2(mp-k)}{mp-k} &= \binom{2k}{k}\binom{(2(m-s)-1)p+p-2t}{(m-s-1)p+p-t} \\ &\equiv \binom{2k}{k}\binom{2(m-s)-1}{m-s-1}\binom{p-2t}{p-t} = 0 \pmod{p}\end{aligned}$$

and for $t>\frac{p}{2}$,

$$\begin{aligned}\binom{2k}{k}\binom{2(mp-k)}{mp-k} &= \binom{(2s+1)p+2t-p}{sp+t}\binom{2(mp-k)}{mp-k} \\ &\equiv \binom{2s+1}{s}\binom{2t-p}{t}\binom{2(mp-k)}{mp-k} = 0 \pmod{p}.\end{aligned}$$

Thus the result is true for $r=1$.

Now assume $p \nmid k$ and $r \geq 2$. Suppose that for $n < r$ and $k \in \{1, 2, \dots, mp^n - 1\}$ we have

$$\binom{2k}{k}\binom{2(mp^n - k)}{mp^n - k} \equiv 0 \pmod{p^n}.$$

When $p \mid s$, by the inductive hypothesis we have

$$\begin{aligned}
&\binom{2s}{s}\binom{2mp^{r-1} - 2s - 2}{mp^{r-1} - s - 1}(2mp^{r-1} - 2s - 1)p \\
&= \frac{s+1}{2(2s+1)}(2mp^{r-1} - 2s - 1)p\binom{2s+2}{s+1}\binom{2mp^{r-1} - 2s - 2}{mp^{r-1} - s - 1} \equiv 0 \pmod{p^r}.
\end{aligned}$$

When $p \nmid s$, by the inductive hypothesis we obtain

$$\begin{aligned}
&\binom{2s}{s}\binom{2mp^{r-1} - 2s - 2}{mp^{r-1} - s - 1}(2mp^{r-1} - 2s - 1)p \\
&= \frac{mp^{r-1} - s}{2}p\binom{2s}{s}\binom{2mp^{r-1} - 2s}{mp^{r-1} - s} \equiv 0 \pmod{p^r}.
\end{aligned}$$

Suppose $k \in \{1, 2, \dots, mp^r - 1\}$. For $t < \frac{p}{2}$, from the above we see that

$$\begin{aligned}
\binom{2k}{k}\binom{2(mp^r - k)}{mp^r - k} &= \binom{2sp + 2t}{sp + t}\binom{(2mp^{r-1} - 2s - 1)p + p - 2t}{(mp^{r-1} - s - 1)p + p - t} \\
&= \binom{2s}{s}\binom{2mp^{r-1} - 2s - 2}{mp^{r-1} - s - 1}(2mp^{r-1} - 2s - 1)pQ \equiv 0 \pmod{p^r},
\end{aligned}$$

and for $t > \frac{p}{2}$,

$$\begin{aligned}
\binom{2k}{k}\binom{2(mp^r - k)}{mp^r - k} &= \binom{(2s+1)p + 2t - p}{sp + t}\binom{(2mp^{r-1} - 2s - 2)p + 2p - 2t}{(mp^{r-1} - s - 1)p + p - t} \\
&\equiv -Q\binom{2s}{s}\binom{2mp^{r-1} - 2s - 2}{mp^{r-1} - s - 1}(2mp^{r-1} - 2s - 1)p \\
&\equiv 0 \pmod{p^r},
\end{aligned}$$

where

$$Q = \sum_{\substack{i=1 \\ p \nmid i}}^{2sp+2t} i \sum_{\substack{i=1 \\ p \nmid i}}^{2(mp^r - sp - t)} i \Big/ \Big(\sum_{\substack{i=1 \\ p \nmid i}}^{sp+t} i^2 \sum_{\substack{i=1 \\ p \nmid i}}^{mp^r - sp - t} i^2 \Big) \in \mathbb{Z}_p.$$

Hence the result is true for $n = r$. Summarizing the above we have proved the result in the case $p \nmid k$.

Now we assume $p \mid k$. Set $l = \mathrm{ord}_p k$ and $k = p^l k_0$. Then $k_0 \in \{1, \dots, mp^{r-l} - 1\}$ and $p \nmid k_0$. For $l \geq r$ obviously we have $k\binom{2k}{k}\binom{2(mp^r-k)}{mp^r-k} \equiv 0 \pmod{p^r}$. For $1 \leq l \leq r - 1$, since $p \nmid k_0$, from the above we deduce that

$$k\binom{2k}{k}\binom{2(mp^r - k)}{mp^r - k} = Wp^l\binom{2k_0}{k_0}\binom{2mp^{r-l} - 2k_0}{mp^{r-l} - k_0} \equiv 0 \pmod{p^r},$$

where $W \in \mathbb{Z}_p$. The proof is now complete.

Lemma 2.13. *Let p be an odd prime, $r, m \in \mathbb{Z}^+$ and $s \in \{0, 1, \dots, mp^{r-1} - 1\}$. Then*

$$\binom{2sp+p-1}{sp+\frac{p-1}{2}}\binom{2(mp^{r-1}-s-1)p+p-1}{(mp^{r-1}-s-1)p+\frac{p-1}{2}} \equiv \binom{2s}{s}\binom{2(mp^{r-1}-s-1)}{mp^{r-1}-s-1} \pmod{p^r}.$$

Proof. For $n \in \mathbb{Z}^+$ and $k \in \{0, 1, \dots, n-1\}$ it is easily seen that

$$\frac{\binom{2n}{n}\binom{n-1}{k}^2}{\binom{2n-1}{2k+1}} = \frac{(2n)!(2k+1)!(2n-2k-2)!(n-1)!^2}{n!^2(2n-1)!k!^2(n-1-k)!^2} = \frac{2(2k+1)}{n} \cdot \frac{(2k)!(2n-2-2k)!}{k!^2(n-1-k)!^2}.$$

Hence,

$$\binom{2k}{k}\binom{2(n-1-k)}{n-1-k} = \frac{n}{2(2k+1)}\frac{\binom{2n}{n}\binom{n-1}{k}^2}{\binom{2n-1}{2k+1}}. \tag{2.4}$$

Using Lemma 2.8 and (2.4) we see that

$$\begin{aligned}
&\binom{2sp+p-1}{sp+\frac{p-1}{2}}\binom{2(mp^{r-1}-s-1)p+p-1}{(mp^{r-1}-s-1)p+(p-1)/2} \\
&= mp^{r-1}\binom{2mp^r}{mp^r}\binom{mp^r-1}{sp+(p-1)/2}^2 \Big/ \Big(2(2s+1)\binom{2mp^r-1}{2sp+p}\Big) \\
&= mp^{r-1}\binom{2mp^{r-1}}{mp^{r-1}}\binom{mp^{r-1}-1}{s}^2 \prod_{\substack{i=1\\ p\nmid i}}^{mp^r}\frac{2mp^r-i}{i}\prod_{\substack{i=1\\ p\nmid i}}^{sp+(p-1)/2}\Big(\frac{mp^r-i}{i}\Big)^2 \\
&\quad\times\Big\{2(2s+1)\binom{2mp^{r-1}-1}{2s+1}\prod_{\substack{i=1\\ p\nmid i}}^{2sp+p}\frac{2mp^r-i}{i}\Big\}^{-1} \\
&= \binom{2s}{s}\binom{2(mp^{r-1}-s-1)}{mp^{r-1}-s-1}\prod_{\substack{i=1\\ p\nmid i}}^{mp^r}\frac{2mp^r-i}{i}\prod_{\substack{i=1\\ p\nmid i}}^{sp+(p-1)/2}\Big(\frac{mp^r-i}{i}\Big)^2 \Big/ \prod_{\substack{i=1\\ p\nmid i}}^{2sp+p}\frac{2mp^r-i}{i} \\
&\equiv \binom{2s}{s}\binom{2(mp^{r-1}-s-1)}{mp^{r-1}-s-1}(-1)^{mp^{r-1}(p-1)}\cdot(-1)^{(2s+1)(p-1)} \\
&= \binom{2s}{s}\binom{2(mp^{r-1}-s-1)}{mp^{r-1}-s-1} \pmod{p^r},
\end{aligned}$$

proving the result.

Lemma 2.14. *Let p be an odd prime, $r, m \in \mathbb{Z}^+$ and $s \in \{0, 1, \dots, mp^{r-1}\}$. Then*

$$\binom{mp^r}{sp}\binom{2sp}{sp}\binom{2mp^r-2sp}{mp^r-sp} \equiv \binom{mp^{r-1}}{s}\binom{2s}{s}\binom{2mp^{r-1}-2s}{mp^{r-1}-s} \pmod{p^{2r}}.$$

Proof. Clearly the result is true for $s = 0$. Now we assume $s \geq 1$. Set $l = \mathrm{ord}_p s$ and

$s = p^l s_0$. By (2.3), (2.4), Lemmas 2.8 and 2.10, we have

$$\begin{aligned}\frac{\binom{2sp}{sp}\binom{2mp^r-2sp}{mp^r-sp}}{\binom{2s}{s}\binom{2mp^{r-1}-2s}{mp^{r-1}-s}} &= \prod_{\substack{i=1\\ p\nmid i}}^{mp^r} \frac{2mp^r-i}{i} \prod_{\substack{i=1\\ p\nmid i}}^{sp} (\frac{mp^r-i}{i})^2 \Big/ \prod_{\substack{i=1\\ p\nmid i}}^{2sp} \frac{2mp^r-i}{i} \\ &\equiv \Big(-2mp^r \sum_{\substack{i=1\\ p\nmid i}}^{mp^r} \frac{1}{i}+1\Big)\Big(-2mp^r \sum_{\substack{i=1\\ p\nmid i}}^{sp} \frac{1}{i}+1\Big)\Big/\Big(-2mp^r \sum_{\substack{i=1\\ p\nmid i}}^{2sp} \frac{1}{i}+1\Big) \\ &\equiv 1 \pmod{p^{r+\min\{l+1,r\}}}.\end{aligned}$$

If $l \ge r-1$, then $r+l+1 \ge 2r$ and so

$$\binom{mp^r}{sp}\binom{2sp}{sp}\binom{2mp^r-2sp}{mp^r-sp} \equiv \binom{mp^{r-1}}{s}\binom{2s}{s}\binom{2mp^{r-1}-2s}{mp^{r-1}-s} \pmod{p^{2r}}.$$

If $0 \le l < r-1$, then $\binom{mp^{r-1}}{s} \equiv 0 \pmod{p^{r-1-l}}$ and so

$$\binom{mp^r}{sp}\binom{2sp}{sp}\binom{2mp^r-2sp}{mp^r-sp} \equiv \binom{mp^{r-1}}{s}\binom{2s}{s}\binom{2mp^{r-1}-2s}{mp^{r-1}-s} \pmod{p^{2r}}.$$

Now the proof is complete.

3. Main results

Theorem 3.1. *Let p be an odd prime and $n \in \mathbb{Z}^+$. Then*

$$S_{np}-S_n \equiv \begin{cases} 8n^2 S_{n-1}(-1)^{\frac{p-1}{2}} p^2 E_{p-3} \pmod{p^3} & \text{if } p>3 \text{ and } p\nmid n, \\ 9(n-1)S_n \pmod{p^3} & \text{if } p=3 \text{ and } 3\nmid n, \\ 0 \pmod{p^{3+\mathrm{ord}_p n}} & \text{if } p\mid n. \end{cases}$$

Proof. Set $r = \mathrm{ord}_p(np)$. Then

$$\begin{aligned} S_{np} &= \sum_{k=0}^{np} \binom{np}{k}\binom{2k}{k}\binom{2(np-k)}{np-k} \\ &= \sum_{s=0}^{n} \binom{np}{sp}\binom{2sp}{sp}\binom{2(n-s)p}{(n-s)p} + \sum_{t=1}^{p-1}\sum_{s=0}^{n-1} \binom{np}{sp+t}\binom{2(sp+t)}{sp+t}\binom{2(np-sp-t)}{np-sp-t}. \end{aligned}$$

If $p>3$ or if $p=3$ and $3\mid n$, using Lemmas 2.3, 2.10 and 2.11 we see that $\binom{np}{sp} \equiv \binom{n}{s}$ $\pmod{p^{r+2}}$ and $\binom{n}{s}\binom{2sp}{sp}\binom{2(n-s)p}{(n-s)p} \equiv \binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s} \pmod{p^{r+2}}$. Thus,

$$\begin{aligned} &\sum_{s=0}^{n} \binom{np}{sp}\binom{2sp}{sp}\binom{2(n-s)p}{(n-s)p} \\ &\equiv \sum_{s=0}^{n} \binom{n}{s}\binom{2sp}{sp}\binom{2(n-s)p}{(n-s)p} \equiv \sum_{s=0}^{n} \binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s} = S_n \pmod{p^{r+2}}. \end{aligned}$$

Hence

$$S_{np}-S_n \equiv \sum_{t=1}^{p-1}\sum_{s=0}^{n-1}\frac{np}{sp+t}\binom{(n-1)p+p-1}{sp+t-1}\binom{2sp+2t}{sp+t}\binom{2(n-1-s)p+2(p-t)}{(n-1-s)p+p-t} \pmod{p^{r+2}}.$$

For $t\in\{1,2,\dots,\frac{p-1}{2}\}$ we have $\frac{p}{2}<p-t<p$. By Lemma 2.6,

$$\binom{2(n-1-s)p+2(p-t)}{(n-1-s)p+p-t}\equiv(2(n-1-s)+1)\binom{2(n-1-s)}{n-1-s}\binom{2(p-t)}{p-t} \pmod{p^2}.$$

By Lemma 2.2, $\binom{2sp+2t}{sp+t}\equiv\binom{2s}{s}\binom{2t}{t} \pmod{p}$. Thus, applying Lemma 2.4 we see that

$$\begin{aligned}&\binom{2sp+2t}{sp+t}\binom{2(n-1-s)p+2(p-t)}{(n-1-s)p+p-t}\\ &\equiv\binom{2s}{s}\binom{2t}{t}(2(n-1-s)+1)\binom{2(n-1-s)}{n-1-s}\binom{2(p-t)}{p-t}\\ &\equiv-(2(n-1-s)+1)\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\frac{2p}{t} \pmod{p^2}.\end{aligned}$$

For $t\in\{\frac{p+1}{2},\dots,p-1\}$ we have $1\le p-t<\frac{p}{2}$. By Lemma 2.6,

$$\binom{2sp+2t}{sp+t}\equiv(2s+1)\binom{2s}{s}\binom{2t}{t} \pmod{p^2}.$$

By Lemma 2.2,

$$\binom{2(n-1-s)p+2(p-t)}{(n-1-s)p+p-t}\equiv\binom{2(n-1-s)}{n-1-s}\binom{2(p-t)}{p-t} \pmod{p}.$$

Thus, applying Lemma 2.4 we get

$$\begin{aligned}&\binom{2sp+2t}{sp+t}\binom{2(n-1-s)p+2(p-t)}{(n-1-s)p+p-t}\\ &\equiv(2s+1)\binom{2s}{s}\binom{2t}{t}\binom{2(n-1-s)}{n-1-s}\binom{2(p-t)}{p-t}\\ &\equiv(2s+1)\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\frac{2p}{t} \pmod{p^2}.\end{aligned}$$

Hence

$$\begin{aligned}&S_{np}-S_n\\ &\equiv\sum_{t=1}^{(p-1)/2}\sum_{s=0}^{n-1}\frac{np}{sp+t}\binom{(n-1)p+p-1}{sp+t-1}\binom{2sp+2t}{sp+t}\binom{2(n-1-s)p+2(p-t)}{(n-1-s)p+p-t}\\ &\quad+\sum_{t=(p+1)/2}^{p-1}\sum_{s=0}^{n-1}\frac{np}{sp+t}\binom{(n-1)p+p-1}{sp+t-1}\binom{2sp+2t}{sp+t}\binom{2(n-1-s)p+2(p-t)}{(n-1-s)p+p-t}\end{aligned}$$

$$\begin{aligned}
&\equiv -\sum_{t=1}^{(p-1)/2}\sum_{s=0}^{n-1}\frac{np}{sp+t}\binom{(n-1)p+p-1}{sp+t-1}(2(n-1-s)+1)\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\frac{2p}{t}\\
&\quad+\sum_{t=(p+1)/2}^{p-1}\sum_{s=0}^{n-1}\frac{np}{sp+t}\binom{(n-1)p+p-1}{sp+t-1}(2s+1)\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\frac{2p}{t}\\
&\equiv -\sum_{t=1}^{(p-1)/2}\sum_{s=0}^{n-1}\frac{np}{t}\binom{n-1}{s}\binom{p-1}{t-1}(2(n-1-s)+1)\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\frac{2p}{t}\\
&\quad+\sum_{t=(p+1)/2}^{p-1}\sum_{s=0}^{n-1}\frac{np}{t}\binom{n-1}{s}\binom{p-1}{t-1}(2s+1)\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\frac{2p}{t}\\
&\equiv 2np^2\sum_{s=0}^{n-1}(2(n-1-s)+1)\binom{n-1}{s}\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\sum_{t=1}^{(p-1)/2}\frac{(-1)^t}{t^2}\\
&\quad-2np^2\sum_{s=0}^{n-1}(2s+1)\binom{n-1}{s}\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\sum_{t=(p+1)/2}^{p-1}\frac{(-1)^t}{t^2}\\
&\equiv 2np^2\sum_{s=0}^{n-1}(2s+1)\binom{n-1}{s}\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\\
&\quad\times\Big(\sum_{t=1}^{(p-1)/2}\frac{(-1)^t}{t^2}-\sum_{t=(p+1)/2}^{p-1}\frac{(-1)^t}{t^2}\Big)\pmod{p^{r+2}}.
\end{aligned}$$

By Lemma 2.7,

$$\begin{aligned}
&\sum_{s=0}^{n-1}(2s+1)\binom{n-1}{s}\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\\
&=S_{n-1}+2\sum_{s=1}^{n-1}s\binom{n-1}{s}\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\\
&=S_{n-1}+2(n-1)\sum_{s=1}^{n-1}\binom{n-2}{s-1}\binom{2s}{s}\binom{2(n-1-s)}{n-1-s}\\
&=S_{n-1}+(n-1)S_{n-1}=nS_{n-1}.
\end{aligned}$$

Note that $B_{p-2}=0$ for $p>3$ and $E_{2n}=2^{2n}E_{2n}(\frac{1}{2})$. From Lemma 2.5 we see that

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\equiv\begin{cases}\frac{1}{2}(-1)^{\frac{p-1}{2}}E_{p-3}\left(\frac{1}{2}\right)\equiv 2(-1)^{\frac{p-1}{2}}E_{p-3}\pmod p & \text{if } p>3,\\ 2\pmod p & \text{if } p=3.\end{cases}$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}-\sum_{k=(p+1)/2}^{p-1}\frac{(-1)^k}{k^2}&=\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}-\sum_{k=1}^{(p-1)/2}\frac{(-1)^{p-k}}{(p-k)^2}\equiv 2\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\\
&\equiv\begin{cases}4(-1)^{\frac{p-1}{2}}E_{p-3}\pmod p & \text{if } p>3,\\ 1\pmod p & \text{if } p=3.\end{cases}
\end{aligned}$$

Now from the above we deduce that $S_{np}-S_n\equiv 2np^2\cdot nS_{n-1}\cdot 4(-1)^{\frac{p-1}{2}}E_{p-3}\pmod{p^{r+2}}$. This yields the result in this case.

Now assume $3\nmid n$. By Lemmas 2.9 and 2.11,

$$\binom{3n}{3s}\equiv\binom{n}{s}(1+9ns^2-9s)\pmod{27},$$

and

$$\begin{aligned}\binom{3n}{3s}\binom{6s}{3s}\binom{6(n-s)}{3(n-s)}&\equiv\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}(1+9n)(1+9ns^2-9s)\\&\equiv\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}(1+9n+9ns^2-9s)\pmod{27}.\end{aligned}$$

Thus,

$$\sum_{s=0}^{n}\binom{3n}{3s}\binom{6s}{3s}\binom{6(n-s)}{3(n-s)}\equiv(1+9n)S_n+9\sum_{s=0}^{n}(ns^2-s)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}\pmod{27}$$

and so

$$\begin{aligned}&S_{3n}-S_n\\&\equiv\sum_{t=1}^{2}\sum_{s=0}^{n-1}\frac{3n}{3s+t}\binom{3(n-1)+3-1}{3s+t-1}\binom{6s+2t}{3s+t}\binom{6(n-1-s)+2(3-t)}{3(n-1-s)+3-t}\\&\quad+9nS_n+9\sum_{s=0}^{n}(ns^2-s)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}\\&\equiv 2n3^2nS_{n-1}(-5/4)+9nS_n+9\sum_{s=0}^{n}(ns^2-s)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}\\&\equiv 9nS_n-9S_{n-1}+9\sum_{s=0}^{n}(ns^2-s)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}\pmod{27}.\end{aligned}$$

By (1.3), for $n\equiv 2\pmod 3$ we have

$$S_n+S_{n-1}\equiv 4(3n(n+1)+1)S_n-32n^2S_{n-1}=(n+1)^2S_{n+1}\equiv 0\pmod 3,$$

for $n\equiv 1\pmod 3$ we have

$$S_n-S_{n-1}\equiv n^2S_n-4(3n(n-1)+1)S_{n-1}=-32(n-1)^2S_{n-2}\equiv 0\pmod 3.$$

Thus,

$$S_n\equiv\left(\frac{n}{3}\right)S_{n-1}\pmod 3\quad\text{for}\quad n\not\equiv 0\pmod 3.\tag{3.1}$$

Applying (3.1) and (2.2) we have

$$S_{3n}-S_n\equiv 9(nS_n-S_{n-1})+9\sum_{s=0}^{n}(ns^2-s)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}$$

$$
\begin{aligned}
&\equiv 9\sum_{s=0}^{n}(ns^2-s)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}\\
&=9\sum_{s=0}^{n}\Big(ns(s-1)+\frac{n-1}{2}(2s-n)+\frac{n(n-1)}{2}\Big)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}\\
&=9n\sum_{s=0}^{n}s(s-1)\binom{n}{s}\binom{2s}{s}\binom{2(n-s)}{n-s}+\frac{9n(n-1)}{2}S_n \pmod{27}.
\end{aligned}
$$

If $s=3k+2$ for some nonnegative integer k, using Lemma 2.2 we find that $\binom{2s}{s}=\binom{3(2k+1)+1}{3k+2}\equiv\binom{2k+1}{k}\binom{1}{2}=0 \pmod 3$. Thus, $3\mid s(s-1)\binom{2s}{s}$ for any nonnegative integer s. Hence, from the above we deduce that

$$S_{3n}-S_n\equiv\frac{9n(n-1)}{2}S_n\equiv 9(n-n^2)S_n\equiv 9(n-1)S_n \pmod{27}.$$

This completes the proof.

Corollary 3.1. *Let $p>3$ be a prime. Then*

$$
\begin{aligned}
S_p&\equiv 4+8(-1)^{\frac{p-1}{2}}p^2E_{p-3} \pmod{p^3},\\
S_{2p}&\equiv 20+128(-1)^{\frac{p-1}{2}}p^2E_{p-3} \pmod{p^3},\\
S_{3p}&\equiv 112+1440(-1)^{\frac{p-1}{2}}p^2E_{p-3} \pmod{p^3}.
\end{aligned}
$$

Remark 3.1. Let p be an odd prime and $m,r\in\mathbb{Z}^+$. Since

$$
\begin{aligned}
S_{mp^r}=&\sum_{s=0}^{mp^{r-1}}\binom{mp^r}{sp}\binom{2sp}{sp}\binom{2mp^r-2sp}{mp^r-sp}\\
&+\sum_{s=0}^{mp^{r-1}-1}\sum_{t=1}^{p-1}\binom{mp^r}{sp+t}\binom{2sp+2t}{sp+t}\binom{2mp^r-2sp-2t}{mp^r-sp-t},
\end{aligned}
$$

applying Lemmas 2.12 and 2.14 we obtain

$$S_{mp^r}\equiv\sum_{s=0}^{mp^{r-1}}\binom{mp^{r-1}}{s}\binom{2s}{s}\binom{2mp^{r-1}-2s}{mp^{r-1}-s}=S_{mp^{r-1}} \pmod{p^{2r}}.$$

This proves (1.11).

Lemma 3.1. *Let $n\in\mathbb{Z}^+$. Then*

$$S_{n+1}\equiv 4(n+1)S_n \pmod{n^2}.$$

Proof. By (1.3),

$$(n+1)^2S_{n+1}=4(3n(n+1)+1)S_n-32n^2S_{n-1}.$$

Thus,

$$(1+2n)S_{n+1}\equiv(n+1)^2S_{n+1}\equiv 4(3n+1)S_n \pmod{n^2}$$

and so

$$S_{n+1}\equiv\frac{4(1+3n)}{1+2n}S_n\equiv 4(1+3n)(1-2n)S_n\equiv 4(1+n)S_n \pmod{n^2}$$

as asserted.

Theorem 3.2. *Let p be an odd prime, and $m, r \in \mathbb{Z}^+$. Then*

$$S_{mp^r+1} \equiv 4(mp^r+1)S_{mp^{r-1}} \pmod{p^{2r}}.$$

Proof. As $\mathrm{ord}_p(m^2p^{2r}) \geq 2r$, from Lemma 3.1 and Remark 3.1 we see that

$$S_{mp^r+1} \equiv 4(mp^r+1)S_{mp^r} \equiv 4(mp^r+1)S_{mp^{r-1}} \pmod{p^{2r}}.$$

This completes the proof.

Theorem 3.3. *Let p be an odd prime and $m, r \in \mathbb{Z}^+$. Then*

$$S_{mp^r-1} \equiv (-1)^{\frac{p-1}{2}} S_{mp^{r-1}-1} \pmod{p^r}.$$

Proof. It is clear that

$$\begin{aligned} S_{mp^r-1} = &\sum_{s=0}^{mp^{r-1}-1} \sum_{\substack{t=0 \\ t\neq(p-1)/2}}^{p-1} \binom{2sp+2t}{sp+t}\binom{mp^r-1}{sp+t}\binom{2(mp^r-1-sp-t)}{mp^r-1-sp-t} \\ &+ \sum_{s=0}^{mp^{r-1}-1} \binom{2sp+p-1}{sp+\frac{p-1}{2}}\binom{mp^r-1}{sp+\frac{p-1}{2}}\binom{2(mp^r-1-sp-\frac{p-1}{2})}{mp^r-1-sp-\frac{p-1}{2}}. \end{aligned}$$

Using Lemma 2.8 we see that

$$\binom{mp^r-1}{sp+t} \equiv \binom{mp^{r-1}-1}{s}(-1)^t \pmod{p^r}.$$

For $t \neq \frac{p-1}{2}$ applying Lemma 2.12 we obtain

$$\begin{aligned} &\binom{2sp+2t}{sp+t}\binom{2(mp^r-1-sp-t)}{mp^r-1-sp-t} \\ &= \binom{2sp+2t}{sp+t}\binom{2(mp^r-sp-t)}{mp^r-sp-t} \cdot \frac{(mp^r-sp-t)^2}{(2mp^r-1-2sp-2t)2(mp^r-sp-t)} \\ &\equiv \binom{2sp+2t}{sp+t}\binom{2(mp^r-sp-t)}{mp^r-sp-t} \cdot \frac{sp+t}{2(2sp+2t+1)} \equiv 0 \pmod{p^r}. \end{aligned}$$

For $t = \frac{p-1}{2}$ using Lemma 2.13 we deduce that

$$\begin{aligned} S_{mp^r-1} &\equiv (-1)^{\frac{p-1}{2}} \sum_{s=0}^{mp^{r-1}-1} \binom{2s}{s}\binom{mp^{r-1}-1}{s}\binom{2(mp^{r-1}-1-s)}{mp^{r-1}-1-s} \\ &= (-1)^{\frac{p-1}{2}} S_{mp^{r-1}-1} \pmod{p^r}. \end{aligned}$$

So the theorem is proved.

Corollary 3.2. *Let p be an odd prime and $m, r \in \mathbb{Z}^+$. Then*

$$P_{mp^r-1} \equiv (-1)^{\frac{p-1}{2}} P_{mp^{r-1}-1} \pmod{p^r}.$$

Proof. From Theorem 3.3, Euler's Theorem and the fact $P_n = 2^n S_n$ we obtain the result.

Theorem 3.4. *Let p be an odd prime and $n \in \mathbb{Z}^+$. Then*

$$S_{np+1} \equiv \begin{cases} (4+12n-9n^2)S_n \pmod{p^3} & \text{if } p=3, \\ 4(np+1)S_n + 32n^2 S_{n-1}(-1)^{\frac{p-1}{2}}(E_{p-3}-1)p^2 \pmod{p^3} & \text{if } p>3. \end{cases}$$

Proof. By (1.3),

$$(np+1)^2 S_{np+1} = 4(3np(np+1)+1)S_{np} - 32n^2p^2 S_{np-1}.$$

Thus, applying Theorems 3.1 and 3.3 we see that for $p>3$,

$$\begin{aligned}(np+1)^2 S_{np+1} &\equiv 4(3n^2p^2+3np+1)(S_n + 8n^2 S_{n-1}(-1)^{\frac{p-1}{2}}p^2E_{p-3}) - 32n^2p^2(-1)^{\frac{p-1}{2}}S_{n-1} \\ &\equiv 4(3n^2p^2+3np+1)S_n + 32n^2S_{n-1}(-1)^{\frac{p-1}{2}}(E_{p-3}-1)p^2 \pmod{p^3},\end{aligned}$$

and for $p=3$,

$$\begin{aligned}(3n+1)^2 S_{3n+1} &= 4(9n(3n+1)+1)S_{3n} - 32n^2\cdot 9S_{3n-1} \\ &\equiv 4(9n+1)(1-9n(n-1))S_n - 9n^2S_{n-1} \\ &\equiv 4(1-9n(n+1))S_n - 9n^2 S_{n-1} \pmod{27}.\end{aligned} \tag{3.2}$$

Since

$$\frac{1}{(np+1)^2} = \frac{(n^2p^2-np+1)^2}{\big((np)^3+1\big)^2} \equiv (n^2p^2-np+1)^2 \equiv 3n^2p^2-2np+1 \pmod{p^3}, \tag{3.3}$$

from the above we deduce that for $p>3$,

$$\begin{aligned}S_{np+1} &\equiv \frac{4(3n^2p^2+3np+1)S_n + 32n^2S_{n-1}(-1)^{\frac{p-1}{2}}(E_{p-3}-1)p^2}{(np+1)^2} \\ &\equiv 4(S_n + 3npS_n + n^2p^2(3S_n + 8S_{n-1}(-1)^{\frac{p-1}{2}}(E_{p-3}-1)))(3n^2p^2-2np+1) \\ &\equiv 4(np+1)S_n + 32n^2S_{n-1}(-1)^{\frac{p-1}{2}}(E_{p-3}-1)p^2 \pmod{p^3}.\end{aligned}$$

Now assume $p=3$. If $3 \mid n$, from (3.2) and (3.3) we deduce that

$$S_{3n+1} \equiv \frac{4S_n}{(3n+1)^2} \equiv 4(1-6n)S_n \equiv (4+12n-9n^2)S_n \pmod{27}.$$

If $3 \nmid n$, then $S_{n-1} \equiv (\frac{n}{3})S_n \pmod 3$ by (3.1). Hence, from (3.2) and (3.3) we deduce that

$$\begin{aligned}S_{3n+1} &\equiv \frac{4S_n - 9n((n+1)S_n + nS_{n-1})}{(3n+1)^2} \equiv \frac{4S_n - 9(n+1+(\frac{n}{3}))S_n}{(3n+1)^2} \\ &\equiv (4-9(2n+1))S_n(1-6n) \equiv (12n-5)S_n \equiv (4+12n-9n^2)S_n \pmod{27}.\end{aligned}$$

Summarizing the above proves the theorem.

Corollary 3.3. *Let $p>3$ be a prime. Then*

$$S_{p+1} \equiv 16 + 16p + 32(-1)^{\frac{p-1}{2}}(E_{p-3}-1)p^2 \pmod{p^3}.$$

Proof. Taking $n=1$ in Theorem 3.4 we obtain the result.

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