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Identities and congruences for Catalan-Larcombe-French numbers

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Abstract

Let $\{P_n\}$ be the Catalan-Larcombe-French numbers given by $P_0 = 1,\ P_1 = 8$ and $(n+1)^2P_{n+1} = 8(3n^2+3n+1)P_n - 128n^2P_{n-1}$ $(n \ge 1)$, and let $S_n = P_n/2^n$. In this paper we obtain some identities and congruences involving $\{S_n\}$. In particular, we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{m^k} \pmod p$ for $m = 7, 16, 25, 32, 64, 160, 800, 1600, 156832$, where p is an odd prime such that $p \nmid m$.

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1. Introduction

Let $[x]$ be the greatest integer not exceeding x. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For positive integers a, b and n, if $n = ax^2 + by^2$ for some integers x and y, we briefly write that $n = ax^2 + by^2$.

Let $\{P_n\}$ be the sequence given by

$$(1.1)\quad P_0 = 1,\ P_1 = 8 \quad \text{and} \quad (n+1)^2P_{n+1} = 8(3n^2+3n+1)P_n - 128n^2P_{n-1}\ (n \ge 1).$$

The numbers P_n are called Catalan-Larcombe-French numbers since Catalan first defined P_n in [2], and in [9] Larcombe and French proved that

$$(1.2)\qquad P_n = 2^n\sum_{k=0}^{[n/2]}(-4)^k\binom{2n-2k}{n-k}^2\binom{n-k}{k} = \sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{2n-2k}{n-k}^2}{\binom{n}{k}}.$$

The numbers P_n occur in the theory of elliptic integrals, and are related to the arithmetic-geometric-mean. See [9] and A053175 in Sloane's database "The On-Line Encyclopedia of Integer Sequences". For known properties of P_n see also [5,7,8,10].

Let $\{S_n\}$ be defined by

$$S_0 = 1,\ S_1 = 4 \quad \text{and} \quad (n+1)^2 S_{n+1} = 4(3n^2+3n+1)S_n - 32n^2 S_{n-1}\ (n \ge 1). \tag{1.3}$$

Comparing (1.3) with (1.1), we see that

$$S_n = P_n/2^n. \tag{1.4}$$

The first few values of S_n are shown below:

$$S_0 = 1,\ S_1 = 4,\ S_2 = 20,\ S_3 = 112,\ S_4 = 676,\ S_5 = 4304,$$
$$S_6 = 28496,\ S_7 = 194240,\ S_8 = 1353508,\ S_9 = 9593104.$$

In this paper we investigate the properties of S_n instead of P_n since S_n is an Apéry-like sequence. Zagier [18] noted that

$$S_n = \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{n}{2k} 4^{n-2k}. \tag{1.5}$$

As observed by Jovovic in 2003 (see [10]),

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} \quad (n = 0, 1, 2, \ldots). \tag{1.6}$$

Recently the author's brother Z.W. Sun stated that

$$S_n = \frac{1}{(-2)^n} \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{k}{n-k}(-4)^k. \tag{1.7}$$

In [17] Z.W. Sun introduced

$$S_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} x^k \quad (n = 0, 1, 2, \ldots)$$

and used it to establish new series for $1/\pi$. Note that $S_n(1) = S_n = P_n/2^n$. In [8], Jarvis and Verrill gave some congruences for $P_n = 2^n S_n$. In Section 2 we establish some new identities involving S_n. For example,

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k \frac{S_k}{8^k} = \frac{S_n}{8^n} \quad \text{and} \quad \sum_{k=0}^{2n} \binom{2n}{k}\binom{2n+k}{k}(-8)^{2n-k} S_k = (-1)^n \binom{2n}{n}^3. \tag{1.8}$$

Let p be an odd prime, $n \in \mathbb{Z}_p$ and $n \not\equiv 0, -16 \pmod p$. We prove that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{(n+16)^k} \equiv \Big(\frac{n(n+16)}{p}\Big) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \pmod p, \tag{1.9}$$

where $(\frac{a}{p})$ is the Legendre symbol. As consequences we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{m^k} \pmod p$ for $m = 7, 16, 25, 32, 64, 160, 800, 1600,\ 156832$. For instance, for any prime $p > 7$,

$$\sum_{k=0}^{p-1} \binom{2k}{k}\frac{S_k}{7^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1, 2, 4 \pmod 7 \text{ and so } p = x^2 + 7y^2, \\ 0 \pmod p & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

We also pose some conjectures on congruences involving S_n. See Conjectures 2.1-2.3.

2. New properties of $\{S_n\}$

Recall that

$$S_n(x)=\sum_{k=0}^n\binom nk\binom{2k}k\binom{2n-2k}{n-k}x^k\quad(n=0,1,2,\ldots)$$

and $S_n=S_n(1)$. From [6, (6.12)] we know that

$$S_n(-1)=\sum_{k=0}^n\binom nk\binom{2k}k\binom{2n-2k}{n-k}(-1)^k=\begin{cases}0&\text{if } n \text{ is odd},\\ \binom{n}{n/2}^2&\text{if } n \text{ is even}.\end{cases}\tag{2.1}$$

If $\{c_n\}$ is a sequence satisfying

$$\sum_{k=0}^n\binom nk(-1)^kc_k=c_n\quad(n=0,1,2,\ldots),$$

we say that $\{c_n\}$ is an even sequence. In [11,14] the author investigated the properties of even sequences.

Lemma 2.1. *Suppose that $\{c_n\}$ is an even sequence.*

(i) ([14, Theorem 2.3]) *If n is odd, then*

$$\sum_{k=0}^n\binom nk\binom{n+k}k(-1)^kc_k=0.$$

(ii) ([14, Theorems 4.3 and 4.4]) *If p is a prime of the form $4k+3$ and $c_0,c_1,\ldots,c_{\frac{p-1}2}\in\mathbb Z_p$, then*

$$\sum_{k=0}^{(p-1)/2}\binom{2k}k^2\frac{c_k}{16^k}\equiv0\pmod{p^2}\quad\textit{and}\quad\sum_{k=0}^{(p-1)/2}\binom{2k}k\frac{c_k}{2^k}\equiv0\pmod p.$$

Lemma 2.2. *For any nonnegative integer n we have*

$$S_n(-x)=\sum_{k=0}^n\binom nk(-1)^k4^{n-k}S_k(x).$$

Proof. Since $\binom{-1/2}k=\binom{2k}k/(-4)^k$ and $\binom{-x}k=(-1)^k\binom{x+k-1}k$, using Vandermonde's identity [6, (3.1)] we see that for any nonnegative integer m,

$$\begin{aligned}\sum_{k=0}^m\binom mk\binom{2k}k(-1)^k4^{m-k}&=4^m\sum_{k=0}^m\binom m{m-k}\binom{-\frac12}k=4^m\binom{m-\frac12}m\\&=4^m\cdot(-1)^m\binom{-\frac12}m=\binom{2m}m.\end{aligned}$$

Note that $\binom nk\binom kr=\binom nr\binom{n-r}{k-r}$. Applying the above we see that

$$\sum_{k=0}^n\binom nk(-1)^k4^{n-k}S_k(x)=\sum_{k=0}^n\binom nk(-1)^k4^{n-k}\sum_{r=0}^k\binom kr\binom{2r}r\binom{2(k-r)}{k-r}x^r$$

$$= \sum_{r=0}^{n} \binom{2r}{r} x^r \binom{n}{r} \sum_{k=r}^{n} \binom{n-r}{k-r} \binom{2(k-r)}{k-r} (-1)^k 4^{n-k}$$
$$= \sum_{r=0}^{n} \binom{n}{r} \binom{2r}{r} x^r (-1)^r \sum_{s=0}^{n-r} \binom{n-r}{s} \binom{2s}{s} (-1)^s 4^{n-r-s}$$
$$= \sum_{r=0}^{n} \binom{n}{r} \binom{2r}{r} (-x)^r \binom{2n-2r}{n-r} = S_n(-x).$$

This proves the lemma.

Lemma 2.3. *For any nonnegative integer* m *we have*

$$\sum_{k=0}^{m} \binom{m}{k} S_k(x) n^{m-k} = \sum_{k=0}^{m} \binom{m}{k} (-1)^k S_k(-x)(n+4)^{m-k}$$

and so

$$\sum_{k=0}^{m} \binom{m}{k} S_k n^{m-k} = \sum_{k=0}^{[m/2]} \binom{m}{2k} \binom{2k}{k}^2 (n+4)^{m-2k}.$$

Proof. Note that $\binom{m}{k}\binom{k}{r} = \binom{m}{r}\binom{m-r}{k-r}$. By Lemma 2.2,

$$\sum_{k=0}^{m} \binom{m}{k} S_k(x) n^{m-k} = \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \sum_{r=0}^{k} \binom{k}{r} (-1)^r S_r(-x) 4^{k-r}$$
$$= \sum_{r=0}^{m} (-1)^r S_r(-x) \sum_{k=r}^{m} \binom{m}{k} \binom{k}{r} 4^{k-r} n^{m-k}$$
$$= \sum_{r=0}^{m} \binom{m}{r} (-1)^r S_r(-x) n^{m-r} \sum_{k=r}^{m} \binom{m-r}{k-r} \left(\frac{4}{n}\right)^{k-r}$$
$$= \sum_{r=0}^{m} \binom{m}{r} (-1)^r S_r(-x) n^{m-r} \left(1+\frac{4}{n}\right)^{m-r}$$
$$= \sum_{r=0}^{m} \binom{m}{r} (-1)^r S_r(-x)(n+4)^{m-r}.$$

Taking $x = 1$ in the above formula and then applying (2.1) we deduce the remaining result.

Theorem 2.1. *Let* n *be a nonnegative integer. Then*

(i) $$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 4^{n-k} S_k = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \binom{n}{n/2}^2 & \text{if } n \text{ is even,} \end{cases}$$

(ii) $$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{S_k}{8^k} = \frac{S_n}{8^n},$$

(iii) $$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-8)^{n-k} S_k = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \binom{n}{n/2}^3 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Taking $x = 1$ in Lemma 2.2 and then applying (2.1) we deduce part (i). By Lemma 2.3,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}S_k m^{n-k}\\&=\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}^2(m+4)^{n-2k}=(-1)^n\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}^2(-m-4)^{n-2k}\\&=(-1)^n\sum_{k=0}^{n}\binom{n}{k}S_k(-m-8)^{n-k}.\end{aligned}$$

That is,

$$\sum_{k=0}^{n}\binom{n}{k}S_k m^{n-k}=\sum_{k=0}^{n}\binom{n}{k}(-1)^k S_k(m+8)^{n-k}. \tag{2.2}$$

Putting $m = 0$ in (2.2) we obtain part (ii). By (ii), $\{\frac{S_n}{8^n}\}$ is an even sequence. Thus applying Lemma 2.1(i), for odd n we have

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-8)^{n-k}S_k=(-8)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k\frac{S_k}{8^k}=0.$$

Let

$$c_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-8)^{n-k}S_k.$$

By (1.5),

$$c_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-8)^{n-k}\sum_{l=0}^{k}\binom{2l}{l}^2\binom{k}{2l}4^{k-2l}.$$

Set

$$F(n,k,l)=\binom{n}{k}\binom{n+k}{k}(-8)^{n-k}\binom{2l}{l}^2\binom{k}{2l}4^{k-2l}.$$

Then

$$c_n=\sum_{k=0}^{n}\sum_{l=0}^{k}F(n,k,l).$$

By the Maple package DoubleSum (see http://cam.tju.edu.cn/∼hou/soft/ds.html and the method in [4]), we find that for $k, l \in \{0, 1, \dots, n-1\}$,

$$\begin{aligned}&F(n,k,l)+\frac{(n+2)^3}{64(n+1)^3}F(n+2,k,l)\\&=R_1(n,k+1,l)F(n,k+1,l)-R_1(n,k,l)F(n,k,l)\\&\quad+R_2(n,k,l+1)F(n,k,l+1)-R_2(n,k,l)F(n,k,l),\end{aligned}$$

where

$$R_1(n,k,l)=-\frac{2k(k-2l)(2n+3)(n^2+3n+4kl+2k-4l+1)}{(n+2-k)(n+1-k)(n+1)^3}$$

and

$$R_2(n,k,l) = -\frac{16(2n+3)kl^3}{(n+2-k)(n+1-k)(n+1)^3}.$$

Suppose that n is even. Since $R_1(n,0,0) = R_2(n,k,0) = 0$ and $F(n,k,l) = 0$ for $l > \frac{k}{2}$, from the above one deduces that

$$\begin{aligned}
&c_n + \frac{(n+2)^3}{64(n+1)^3} c_{n+2} \\
&= \sum_{k=0}^{n}\sum_{l=0}^{k} F(n,k,l) + \frac{(n+2)^3}{64(n+1)^3} \sum_{k=0}^{n+2}\sum_{l=0}^{k} F(n+2,k,l) \\
&= \sum_{k=0}^{n-1}\sum_{l=0}^{k} \Big(F(n,k,l) + \frac{(n+2)^3}{64(n+1)^3} F(n+2,k,l)\Big) + \sum_{l=0}^{n} F(n,n,l) \\
&\quad + \frac{(n+2)^3}{64(n+1)^3} \sum_{l=0}^{n+2} \big(F(n+2,n+2,l) + F(n+2,n+1,l) + F(n+2,n,l)\big) \\
&= \sum_{l=0}^{n-1}\sum_{k=l}^{n-1} \big(R_1(n,k+1,l)F(n,k+1,l) - R_1(n,k,l)F(n,k,l)\big) \\
&\quad + \sum_{k=0}^{n-1}\sum_{l=0}^{k} \big(R_2(n,k,l+1)F(n,k,l+1) - R_2(n,k,l)F(n,k,l)\big) \\
&\quad + \sum_{l=0}^{n} F(n,n,l) + \frac{(n+2)^3}{64(n+1)^3} \sum_{l=0}^{n+2} \Big\{ \binom{2n+4}{n+2}\binom{2l}{l}^2 \binom{n+2}{2l} 4^{n+2-2l} \\
&\quad + \binom{n+2}{n+1}\binom{2n+3}{n+1}(-8)\binom{2l}{l}^2\binom{n+1}{2l} 4^{n+1-2l} \\
&\quad + \binom{n+2}{n}\binom{2n+2}{n}(-8)^2\binom{2l}{l}^2\binom{n}{2l} 4^{n-2l} \Big\} \\
&= \sum_{l=0}^{n-1} \big(R_1(n,n,l)F(n,n,l) - R_1(n,l,l)F(n,l,l)\big) \\
&\quad + \sum_{k=0}^{n-1} \big(R_2(n,k,k+1)F(n,k,k+1) - R_2(n,k,0)F(n,k,0)\big) + \sum_{l=0}^{n} F(n,n,l) \\
&\quad + \frac{(n+2)^3}{64(n+1)^3} \Big\{ \binom{2n+4}{n+2}\binom{n+2}{(n+2)/2}^2 + \sum_{l=0}^{n/2} (2n+1)\binom{2n}{n}\binom{2l}{l}^2\binom{n}{2l} 4^{n+3-2l} \\
&\quad \times \Big(\frac{2n+3}{(n+1-2l)(n+2-2l)} - \frac{2n+3}{n+1-2l} + (n+1) \Big) \Big\} \\
&= \sum_{l=0}^{n/2} (1 + R_1(n,n,l)) \binom{2n}{n}\binom{2l}{l}^2\binom{n}{2l} 4^{n-2l} \\
&\quad + \frac{(n+2)^3}{64(n+1)^3} \binom{2n}{n} \Big\{ \frac{4(2n+1)(2n+3)}{(n+1)(n+2)} \binom{n+2}{(n+2)/2}^2
\end{aligned}$$

$$+\sum_{l=0}^{n/2}(2n+1)\binom{2l}{l}^2\binom{n}{2l}4^{n+3-2l}\Big(n+1-\frac{2n+3}{n+2-2l}\Big)\Big\}$$

$$=\frac{\binom{2n}{n}}{(n+1)^3}\Big\{\sum_{l=0}^{n/2}\binom{2l}{l}^2\binom{n}{2l}4^{n-2l}\Big((n+1)^3-n(2n+3)(n-2l)(n^2+5n+1+4(n-1)l)$$

$$+(n+2)^3(2n+1)\Big(n+1-\frac{2n+3}{n+2-2l}\Big)\Big)+\frac{(n+2)^2(2n+1)(2n+3)}{16(n+1)}\binom{n+2}{(n+2)/2}^2\Big\}$$

$$=\frac{(2n+3)\binom{2n}{n}}{(n+1)^3}\Big\{\sum_{l=0}^{n/2}\binom{2l}{l}^2\binom{n}{2l}4^{n-2l}\Big(n^3+12n^2+11n+3-(2n^3-14n^2-2n)l$$

$$+(8n^2-8n)l^2-\frac{(n+2)^3(2n+1)}{n+2-2l}\Big)+(n+1)(2n+1)\binom{n}{n/2}^2\Big\}.$$

Using Zeilberger's Maple package EKHAD one can easily prove that for even n,

$$\sum_{l=0}^{n/2}\binom{2l}{l}^2\binom{n}{2l}4^{n-2l}\Big(n^3+12n^2+11n+3-(2n^3-14n^2-2n)l$$

$$+(8n^2-8n)l^2-\frac{(n+2)^3(2n+1)}{n+2-2l}\Big)$$

$$=-(n+1)(2n+1)\binom{n}{n/2}^2.$$

Thus,

$$c_n+\frac{(n+2)^3}{64(n+1)^3}c_{n+2}=0\quad\text{and so}\quad c_{n+2}=-\frac{64(n+1)^3}{(n+2)^3}c_n.$$

Since $c_0=1$ and

$$(-1)^{(n+2)/2}\binom{n+2}{(n+2)/2}^3=-\frac{64(n+1)^3}{(n+2)^3}\cdot(-1)^{n/2}\binom{n}{n/2}^3,$$

we must have $c_n=(-1)^{n/2}\binom{n}{n/2}^3$. Thus part (iii) holds and the proof is complete.

Remark 2.1 By Theorem 2.1(ii), $\{\frac{S_n}{8^n}\}$ is an even sequence. Thus, applying [11, Theorem 4.1] we see that for any function f,

$$\sum_{k=0}^{n}\binom{n}{k}\Big(f(k)-(-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\Big)8^kS_{n-k}=0\quad(n=0,1,2,\ldots).$$

Using [14, Theorem 2.2] we also have

$$\sum_{k=0}^{n}\binom{n}{k}(2n-k)(-8)^kS_{2n-1-k}=0\quad(n=0,1,2,\ldots).$$

Lemma 2.4. *Let p be an odd prime, $u,c_0,c_1,\ldots,c_{p-1}\in\mathbb{Z}_p$ and $u\not\equiv 1\pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^kc_k\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k\pmod p.$$

Proof. Note that $p \mid \binom{2k}{k}$ for $\frac{p}{2} < k < p$, $\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}$ and $\binom{n}{k}\binom{n+k}{k} = \binom{2k}{k}\binom{n+k}{2k}$. Using Fermat's little theorem we deduce that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k c_k\\&\equiv \sum_{k=0}^{(p-1)/2}\binom{2k}{k}c_k u^k(1-u)^{p-1-2k} = \sum_{k=0}^{(p-1)/2}\binom{2k}{k}c_k u^k\sum_{r=0}^{p-1-2k}\binom{p-1-2k}{r}(-u)^r\\&=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}c_k(-1)^{n-k}\binom{p-1-2k}{n-k} = \sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}c_k\binom{n+k-p}{n-k}\\&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}c_k\binom{n+k}{n-k} = \sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k \pmod p.\end{aligned}$$

Thus the lemma is proved.

Lemma 2.5. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv -1 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{8(1+x)^2}\Big)^k S_k \equiv \sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^3\Big(-\frac{x^2}{64}\Big)^k \pmod p.$$

Proof. Taking $u = -x$ and $c_k = \frac{S_k}{(-8)^k}$ in Lemma 2.4 and then applying Theorem 2.1(iii) we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{-x}{(1+x)^2}\Big)^k\frac{S_k}{(-8)^k}\\&\equiv\sum_{n=0}^{p-1}(-x)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{S_k}{(-8)^k} = \sum_{k=0}^{(p-1)/2}\Big(\frac{-x}{-8}\Big)^{2k}(-1)^k\binom{2k}{k}^3 \pmod p.\end{aligned}$$

This yields the result.

Theorem 2.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{32^k} \equiv \begin{cases}(-1)^{\frac{p-1}{2}}4x^2 \pmod p & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p = x^2+2y^2,\\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8.\end{cases}$$

Proof. Taking $x = 1$ in Lemma 2.5 we find that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{32^k} \equiv \sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3\frac{1}{(-64)^k} \pmod p.$$

Now applying [12, Theorems 3.3-3.4] we deduce the result.

Theorem 2.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{16^k} \equiv \begin{cases}4x^2 \pmod p & \text{if } p\equiv 1 \pmod 4 \text{ and so } p = x^2+4y^2,\\ 0 \pmod p & \text{if } p \equiv 3 \pmod 4.\end{cases}$$

Proof. From Theorem 2.1(ii) we know that $\{\frac{S_n}{8^n}\}$ is an even sequence. Thus applying Lemma 2.1(ii) we have $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{16^k}\equiv 0 \pmod p$ for $p\equiv 3 \pmod 4$. Now assume $p\equiv 1 \pmod 4$ and so $p=x^2+4y^2$. Let $t\in\{1,2,\ldots,\frac{p-1}{2}\}$ be given by $t^2\equiv -1 \pmod p$. By Lemma 2.5,

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{16^k}&\equiv\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{t}{8(1+t)^2}\Big)^kS_k\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3\Big(-\frac{t^2}{64}\Big)^k\\&\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3\frac{1}{64^k}\pmod p.\end{aligned}$$

It is well known that (see for example [1])

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3\frac{1}{64^k}\equiv 4x^2-2p\pmod{p^2}.$$

Thus $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{16^k}\equiv 4x^2 \pmod p$, which completes the proof.

Theorem 2.4. *Let p be an odd prime. Then*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{S_k}{128^k}\\&\equiv\begin{cases}8x^3-6xp\pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1\pmod 4 \text{ and } 4\mid x-1,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned}$$

Proof. It is clear that for $k\in\{0,1,\ldots,\frac{p-1}{2}\}$,

$$\begin{aligned}&\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k}\\&=\binom{2k}{k}\binom{\frac{p-1}{2}+k}{2k}=\binom{2k}{k}\frac{(p^2-1^2)(p^2-3^2)\cdots(p^2-(2k-1)^2)}{2^{2k}\cdot(2k)!}\\&\equiv\binom{2k}{k}(-1)^k\frac{1^2\cdot 3^2\cdots(2k-1)^2}{2^{2k}\cdot(2k)!}=\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.\end{aligned}\tag{2.3}$$

Note that $2^{p-1}+1-2(-1)^{\frac{p-1}{4}}2^{\frac{p-1}{2}}=\big((-1)^{\frac{p-1}{4}}2^{\frac{p-1}{2}}-1\big)^2\equiv 0\pmod{p^2}$ for $p\equiv 1\pmod 4$. From Theorem 2.1(iii) and (2.3) we deduce that

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{S_k}{128^k}\\&\equiv\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k}\frac{S_k}{(-8)^k}\\&=\begin{cases}0\pmod{p^2} & \text{if } 4\mid p-3,\\ (-8)^{-\frac{p-1}{2}}(-1)^{\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^3\equiv\big(\frac{2^{p-1}+1}{2}\big)^{-3}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^3\pmod{p^2} & \text{if } 4\mid p-1.\end{cases}\end{aligned}$$

By [3], for $p=x^2+4y^2\equiv 1 \pmod 4$ with $4\mid x-1$,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\equiv \frac{2^{p-1}+1}{2}\Big(2x-\frac{p}{2x}\Big) \pmod{p^2}.$$

Thus,

$$\Big(\frac{2^{p-1}+1}{2}\Big)^{-3}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^3\equiv \Big(2x-\frac{p}{2x}\Big)^3\equiv 8x^3-6xp \pmod{p^2}.$$

Now putting the above together we deduce the result.

For an odd prime p and $a\in\mathbb{Z}_p$ let $\langle a\rangle_p\in\{0,1,\ldots,p-1\}$ be given by $a\equiv\langle a\rangle_p \pmod p$.

Theorem 2.5. *Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $\langle a\rangle_p\equiv 1 \pmod 2$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{S_k}{8^k}\equiv 0 \pmod{p^2}.$$

In particular, for $a=-\frac13,-\frac14,-\frac16$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{216^k}S_k\equiv 0 \pmod{p^2}\quad \textit{for}\quad p\equiv 2 \pmod 3,$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{512^k}S_k\equiv 0 \pmod{p^2}\quad \textit{for}\quad p\equiv 5,7 \pmod 8,$$
$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{3456^k}S_k\equiv 0 \pmod{p^2}\quad \textit{for}\quad p\equiv 3 \pmod 4.$$

Proof. This is immediate from Theorem 2.1(ii) and [13, Theorem 2.4].

Theorem 2.6. *Let p be an odd prime, $n\in\mathbb{Z}_p$ and $n\not\equiv 0,-16 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(n+16)^k}\equiv\Big(\frac{n(n+16)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \pmod p,$$

where $(\frac{a}{p})$ is the Legendre symbol.

Proof. Clearly $p\mid\binom{2k}{k}$ for $\frac p2<k<p$ and $p\mid\binom{2k}{k}\binom{4k}{2k}$ for $\frac p4<k<p$. Note that $\binom{\frac{p-1}{2}}{k}\equiv\binom{-\frac12}{k}=\frac{\binom{2k}{k}}{(-4)^k} \pmod p$ for $0\le k\le\frac{p-1}{2}$. By Lemma 2.3,

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(n+16)^k}\\ &\equiv\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}S_k\Big(\frac{-4}{n+16}\Big)^k\equiv\Big(\frac{-n-16}{p}\Big)\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}S_k\Big(\frac{n+16}{-4}\Big)^{\frac{p-1}{2}-k}\\ &=\Big(\frac{-n-16}{p}\Big)\sum_{k=0}^{[p/4]}\binom{\frac{p-1}{2}}{2k}\binom{2k}{k}^2\Big(-\frac n4\Big)^{\frac{p-1}{2}-2k}\end{aligned}$$

$$\equiv \Big(\frac{-n(-n-16)}{p}\Big)\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}}{(-4)^{2k}}\binom{2k}{k}^2\frac{1}{(-n/4)^{2k}} \equiv \Big(\frac{n(n+16)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \pmod p.$$

This proves the theorem.

Theorem 2.7. *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{7^k} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{25^k}$$
$$\equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and so } p=x^2+7y^2,\\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. Taking $n=\pm 9$ in Theorem 2.6 and then applying [12, Theorem 5.2] we deduce the result.

Theorem 2.8. *Let p be a prime such that $p\equiv 1,7,17,23 \pmod{24}$. Then*

$$\Big(\frac{3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{64^k} \equiv \Big(\frac{6}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-32)^k}$$
$$\equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,7 \pmod{24} \text{ and so } p=x^2+6y^2,\\ 0 \pmod p & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}$$

Proof. Taking $n=\pm 48$ in Theorem 2.6 and then applying [12, Theorem 5.4] we deduce the result.

Theorem 2.9. *Let $p>7$ be a prime. Then*

$$\Big(\frac{2}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{800^k} \equiv \Big(\frac{3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-768)^k}$$
$$\equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+2y^2,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Proof. By [15, Theorem 5.6],

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8\end{cases}$$

Now taking $n=\pm 28^2=\pm 784$ in Theorem 2.6 and then applying the above we obtain the result.

Theorem 2.10. *Let p be a prime such that $p\equiv 1,9 \pmod{10}$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{160^k} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-128)^k}$$
$$\equiv \begin{cases} (\frac{2}{p})4x^2 \pmod p & \text{if } p\equiv 1,9,11,19 \pmod{40} \text{ and so } p=x^2+10y^2,\\ 0 \pmod p & \text{if } p\equiv 21,29,31,39 \pmod{40}.\end{cases}$$

Proof. Taking $n=\pm 144$ in Theorem 2.6 and then applying [12, (5.9)] we deduce the result.

Theorem 2.11. *Let $p>7$ be a prime such that $p\equiv \pm 1 \pmod 8$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{1600^k}\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-1568)^k}$$
$$\equiv \begin{cases} (\frac{-1}{p})4x^2 \pmod p & \text{if } p\equiv 1,3,4,5,9 \pmod{11} \text{ and so } p=x^2+22y^2,\\ 0 \pmod p & \text{if } p\equiv 2,6,7,8,10 \pmod{11}.\end{cases}$$

Proof. Taking $n=\pm 1584$ in Theorem 2.6 and then applying [12, (5.9)] we deduce the result.

Theorem 2.12. *Let p be a prime such that $p\neq 5,7,13$ and $(\frac{p}{29})=1$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{156832^k}\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-156800)^k}$$
$$\equiv \begin{cases} (\frac{2}{p})4x^2 \pmod p & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+58y^2,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Proof. Taking $n=\pm 396^2=\pm 156816$ in Theorem 2.6 and then applying [12, (5.9)] we deduce the result.

Theorem 2.13. *Let p be an odd prime, $n\in\mathbb{Z}_p$ and $n\not\equiv 0 \pmod p$.*

(i) *If $n\not\equiv 4 \pmod p$, then*

$$\sum_{k=0}^{p-1}\frac{S_k(x)}{n^k}\equiv \sum_{k=0}^{p-1}\frac{S_k(-x)}{(4-n)^k} \pmod p \quad \text{and so} \quad \sum_{k=0}^{p-1}\frac{S_k}{n^k}\equiv \sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(4-n)^{2k}} \pmod p.$$

(ii) *If $n\not\equiv 16 \pmod p$, then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k(x)}{n^k}\equiv \Big(\frac{n(n-16)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k(-x)}{(16-n)^k} \pmod p.$$

Proof. Since $\binom{p-1}{k}\equiv(-1)^k \pmod p$ and $\binom{\frac{p-1}{2}}{k}\equiv \binom{2k}{k}/(-4)^k \pmod p$, taking $m=p-1$ and replacing n with $-n$ in Lemma 2.3 we deduce part (i), and taking $m=\frac{p-1}{2}$ and replacing n with $-\frac{n}{4}$ in Lemma 2.3 we deduce part (ii).

The Apéry numbers $\{A_n\}$ and Franel numbers $\{f_n\}$ are given by

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \quad\text{and}\quad f_n=\sum_{k=0}^{n}\binom{n}{k}^3.$$

See A005259 and A000172 in Sloane's database "The On-Line Encyclopedia of Integer Sequences". Let p be an odd prime. In [16] the author posed many conjectures for $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{m^k} \pmod{p^2}$. He also made conjectures on $f_{\frac{p-1}{2}} \pmod{p^2}$ and $f_{\frac{p^r-1}{2}} \pmod{p^r}$. Since $\{S_n\}$ and $\{f_n\}$ are Apéry-like sequences, they should have similar properties. By doing calculations with Maple we pose the following conjectures, which were checked for $p<100$ and $r\le 3$.

Conjecture 2.1. *Let p be an odd prime, $n \in \{\pm 156816, \pm 1584, \pm 784, \pm 144, \pm 48, 16, \pm 9\}$ and $n \not\equiv 0, -16 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{(n+16)^k} \equiv \left(\frac{n(n+16)}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{n^{2k}} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{16^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \pmod{p^4}.$$

Also, for $p > 3$ and $p \equiv 1, 3 \pmod 8$,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{32^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \pmod{p^3}.$$

Conjecture 2.2. *Let p be an odd prime. If $p \equiv 1, 3 \pmod 8$ and so $p = x^2 + 2y^2$, then*

$$S_{\frac{p-1}{2}} \equiv (5 \cdot 2^{p-1} - 1)x^2 - 2p \pmod{p^2}$$

and

$$S_{\frac{p^2-1}{2}} \equiv 4(5 \cdot 2^{p-1} - 1)x^4 - 16x^2 p \pmod{p^2}.$$

Conjecture 2.3. *Let p be an odd prime. If $p \equiv 5, 7 \pmod 8$, then*

$$S_{\frac{p^2-1}{2}} \equiv p^2 \pmod{p^3} \quad \text{and} \quad S_{\frac{p^r-1}{2}} \equiv 0 \pmod{p^r} \quad \text{for} \quad r = 1, 2, 3, \ldots.$$

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