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Congruences involving binomial coefficients and Apéry-like numbers

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Abstract

For $n = 0, 1, 2, \ldots$ let $W_n = \sum_{k=0}^{[n/3]} \binom{2k}{k}\binom{3k}{k}\binom{n}{3k}(-3)^{n-3k}$, where $[x]$ is the greatest integer not exceeding x. Then $\{W_n\}$ is an Apéry-like sequence. In this paper we deduce many congruences involving $\{W_n\}$, in particular we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{m^k} \pmod p$ for $m = -640332, -5292, -972, -108, -44, -27, -12, 8, 54, 243$ by using binary quadratic forms, where $p > 3$ is a prime. We also prove several congruences for generalized Apéry-like numbers, and pose 29 challenging conjectures on congruences involving binomial coefficients and Apéry-like numbers.

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1. Introduction

For $s > 1$ let $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$. In 1979, in order to prove $\zeta(2)$ and $\zeta(3)$ are irrational, Apéry [4] introduced the Apéry numbers $\{A_n\}$ and $\{A'_n\}$ given by

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \quad \text{and} \quad A'_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

It is well known (see [7]) that

$$(n+1)^3A_{n+1} = (2n+1)(17n(n+1)+5)A_n - n^3A_{n-1} \quad (n \ge 1),$$
$$(n+1)^2A'_{n+1} = (11n(n+1)+3)A'_n + n^2A'_{n-1} \quad (n \ge 1).$$

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. The first kind of Apéry-like numbers $\{u_n\}$ satisfies

$$(1.1) \quad u_0 = 1,\ u_1 = b,\ (n+1)^3u_{n+1} = (2n+1)(an(n+1)+b)u_n - cn^3u_{n-1} \quad (n \ge 1),$$

where $a, b, c \in \mathbb{Z}$ and $c \ne 0$. Let $[x]$ be the greatest integer not exceeding x, and let

$$D_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}, \quad T_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2,$$
$$b_n = \sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}\binom{n+k}{k}(-3)^{n-3k}.$$

Then $\{A_n\}$, $\{D_n\}$, $\{b_n\}$ and $\{T_n\}$ are the first kind of Apéry-like numbers with $(a, b, c) = (17, 5, 1), (10, 4, 64), (-7, -3, 81), (12, 4, 16)$, respectively. The numbers $\{D_n\}$ are called

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Domb numbers, and $\{b_n\}$ are called Almkvist-Zudilin numbers. For $\{A_n\}$, $\{D_n\}$, $\{b_n\}$ and $\{T_n\}$ see A005259, A002895, A125143 and A290575 in Sloane's database "The On-Line Encyclopedia of Integer Sequences" [40]. For the congruences involving T_n see the author's recent paper [33].

In 2009, Zagier [41] studied the second kind of Apéry-like numbers $\{u_n\}$ given by

$$(1.2)\qquad u_0=1,\ u_1=b \quad\text{and}\quad (n+1)^2u_{n+1}=(an(n+1)+b)u_n-cn^2u_{n-1}\ (n\ge 1),$$

where $a,b,c\in\mathbb{Z}$ and $c\ne 0$. Let

$$\begin{aligned}
f_n&=\sum_{k=0}^{n}\binom nk^3=\sum_{k=0}^{n}\binom nk^2\binom{2k}{n},\\
S_n&=\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{n}{2k}4^{n-2k}=\sum_{k=0}^{n}\binom nk\binom{2k}{k}\binom{2n-2k}{n-k},\\
a_n&=\sum_{k=0}^{n}\binom nk^2\binom{2k}{k},\quad Q_n=\sum_{k=0}^{n}\binom nk(-8)^{n-k}f_k,\\
W_n&=\sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}(-3)^{n-3k}.
\end{aligned}$$

In [41] Zagier stated that $\{A'_n\}$, $\{f_n\}$, $\{S_n\}$, $\{a_n\}$, $\{Q_n\}$ and $\{W_n\}$ are the second kind of Apéry-like sequences with $(a,b,c)=(11,3,-1),(7,2,-8),(12,4,32),(10,3,9),(-17,-6,72),$ $(-9,-3,27)$, respectively. The sequence $\{f_n\}$ is called Franel numbers. In [14,31,32] the author systematically investigated identities and congruences for sums involving S_n or f_n. For $\{A'_n\}$, $\{f_n\}$, $\{S_n\}$, $\{a_n\}$, $\{Q_n\}$ and $\{W_n\}$, see A005258, A000172, A081085, A002893, A093388 and A291898 in Sloane's database [40].

Apéry-like numbers have fascinating properties and they are concerned with Picard-Fuchs differential equation, modular forms, hypergeometric series, elliptic curves, series for $\frac{1}{\pi}$, supercongruences, binary quadratic forms, combinatorial identities, Bernoulli numbers and Euler numbers. See typical papers [1,2,3,6,8,9,10,11,13,17,18,21,30,35,36,39].

For $a\in\mathbb{Z}$ and given odd prime p let $(\frac ap)$ be the Legendre symbol. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For positive integers a,b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly write that $n=ax^2+by^2$.

In Section 2 we obtain some congruences for sums involving W_n. From [9, (6.4)] we know that

$$(1.3)\qquad \Big(\sum_{k=0}^{\infty}W_kx^k\Big)^2=\frac{1}{1-27x^2}\sum_{k=0}^{\infty}\binom{2k}{k}\Big(\frac{x(1+9x+27x^2)}{(1-27x^2)^2}\Big)^kW_k.$$

We prove the $p-$analogue of (1.3):

$$(1.4)\qquad \Big(\sum_{k=0}^{p-1}W_kx^k\Big)^2\equiv\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x(1+9x+27x^2)}{(1-27x^2)^2}\Big)^kW_k \pmod p,$$

where $p>3$ is a prime, $x\in\mathbb{Z}_p$ and $(x+3)(1+9x+27x^2)(1+9x)(1+27x^2)(1-27x^2)\not\equiv 0 \pmod p$.

Suppose that $p>3$ is a prime, $n\in\mathbb{Z}_p$ and $n(n-12)\not\equiv 0 \pmod p$. In Section 2, we show that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(n-12)^k}\equiv\Big(\frac{n(n-12)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}} \pmod p. \tag{1.5}$$

As consequences, we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{m^k} \pmod p$ for $m=-640332,-5292,-972,-108,$ $-44,-27,8,54,243$ by using binary quadratic forms. We also determine $W_{\frac{p-1}{2}}$, $\sum_{k=0}^{p-1}\frac{W_k}{(-3)^k}$, $\sum_{k=0}^{p-1}\frac{W_k}{(-9)^k}$ and $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-12)^k}$ modulo p.

Let p be an odd prime. In 2010 Jarvis and Verrill [13] discovered relations between u_n and u_{p-1-n} modulo p for $u_n=A'_n,a_n,f_n$ or S_n. For example, they proved $f_n\equiv(-8)^n f_{p-1-n} \pmod p$ for $n=0,1,\ldots,p-1$. In Section 3 we establish a vast generalization of such congruences for generalized Apéry-like numbers $\{u_n\}$. See Theorem 3.1.

In [20] S. Ramanujan made some conjectures for $1/\pi$, which involve the following four sums

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\ \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},\ \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\ \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

The author's brother Z.W. Sun in [39] and the author in [25] posed many conjectures on congruences for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k} \pmod{p^2}, \tag{1.6}$$

where $m\in\mathbb{Z}$ and p is an odd prime with $p\nmid m$. Some of such conjectures were proved by the author in [27-29]. In particular, most of conjectures were solved when the modulus is p. Let $p>3$ be a prime. Recently Liu [16] conjectured congruences for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k} \pmod{p^3}$$

in terms of $p-$adic gamma functions. Based on calculations with Maple, in Section 4 we pose 29 challenging conjectures on congruences involving sums in (1.6) or Apéry-like numbers. See Conjectures 4.1-4.29.

2. Congruences for sums involving W_n

For any nonnegative integer n, define

$$W_n(x)=\sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}x^{n-3k}.$$

Then $W_n=W_n(-3)$. In this section we establish some congruences involving W_n and $W_n(x)$ modulo a prime. We begin with three useful lemmas.

Lemma 2.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}W_k(x)y^{n-k}=W_n(x+y).$$

Proof. It is clear that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}W_k(x)y^{n-k}&=\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\sum_{r=0}^{k}\binom{2r}{r}\binom{3r}{r}\binom{k}{3r}x^{k-3r}\\&=\sum_{r=0}^{n}\binom{2r}{r}\binom{3r}{r}y^{n-3r}\sum_{k=r}^{n}\binom{n}{k}\binom{k}{3r}\Big(\frac{x}{y}\Big)^{k-3r}\\&=\sum_{r=0}^{n}\binom{2r}{r}\binom{3r}{r}y^{n-3r}\sum_{k=3r}^{n}\binom{n}{3r}\binom{n-3r}{k-3r}\Big(\frac{x}{y}\Big)^{k-3r}\\&=\sum_{r=0}^{n}\binom{2r}{r}\binom{3r}{r}\binom{n}{3r}y^{n-3r}\Big(1+\frac{x}{y}\Big)^{n-3r}=W_n(x+y).\end{aligned}$$

This proves the lemma.

Let $\{P_n(x)\}$ be the famous Legendre polynomials given by

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(\frac{x-1}{2}\Big)^k=\frac{1}{2^n}\sum_{k=0}^{[n/2]}\binom{n}{k}(-1)^k\binom{2n-2k}{n}x^{n-2k}.$$

It is well known that

$$P_0(x)=1,\ P_1(x)=x,\ (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)\quad(n\ge 1).$$

Lemma 2.2. *Suppose that $p>3$ is a prime and $m,x\in\mathbb{Z}_p$ with $mx\not\equiv 0\pmod p$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{W_k(x+m)}{m^k}&\equiv W_{p-1}(x)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-x)^{3k}}\equiv P_{[\frac{p}{3}]}\Big(1+\frac{54}{x^3}\Big)\\&\equiv-\Big(\frac{p}{3}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-3x(x^3-216)n-2x^6-1080x^3+108^2}{p}\Big)\pmod p.\end{aligned}$$

Proof. Since $\binom{p-1}{k}\equiv(-1)^k\pmod p$, using Lemma 2.1 and Fermat's little theorem we see that

$$\sum_{k=0}^{p-1}\frac{W_k(x+m)}{m^k}\equiv\sum_{k=0}^{p-1}\binom{p-1}{k}W_k(x+m)(-m)^{p-1-k}=W_{p-1}(x)\pmod p.$$

On the other hand, since $p\mid\binom{2k}{k}\binom{3k}{k}$ for $\frac{p}{3}<k<p$ we have

$$W_{p-1}(x)=\sum_{k=0}^{[\frac{p-1}{3}]}\binom{2k}{k}\binom{3k}{k}\binom{p-1}{3k}x^{p-1-3k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-x)^{3k}}\pmod p.$$

By [28, Corollary 3.1],

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-x)^{3k}}\equiv P_{[\frac p3]}\Big(1+\frac{54}{x^3}\Big)\equiv -\Big(\frac p3\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-3(1-\frac{216}{x^3})n+\frac{108^2}{x^6}-\frac{1080}{x^3}-2}{p}\Big)$$
$$=-\Big(\frac p3\Big)\sum_{n=0}^{p-1}\Big(\frac{(\frac{n}{x^2})^3-3(1-\frac{216}{x^3})\frac{n}{x^2}+\frac{108^2}{x^6}-\frac{1080}{x^3}-2}{p}\Big)$$
$$=-\Big(\frac p3\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-3x(x^3-216)n-2x^6-1080x^3+108^2}{p}\Big)\pmod p.$$

Thus the lemma is proved.

Let $p>3$ be a prime. Taking $m=1$ and $x=-4$ in Lemma 2.2 yields

$$\sum_{k=0}^{p-1}W_k\equiv -\Big(\frac{-6}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-840n+9074}{p}\Big)\pmod p. \tag{2.1}$$

Taking $m=-1$ and $x=-2$ in Lemma 2.2 yields

$$\sum_{k=0}^{p-1}(-1)^kW_k\equiv -\Big(\frac{-6}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-336n+2522}{p}\Big)\pmod p. \tag{2.2}$$

Lemma 2.3. *For any nonnegative integer n we have*

$$\sum_{k=0}^{n}\binom nk W_k3^{n-k}=\begin{cases}\binom{2n/3}{n/3}\binom{n}{n/3} & \text{if } 3\mid n,\\ 0 & \text{if } 3\nmid n.\end{cases}$$

Proof. Putting $x=-3$ and $y=3$ in Lemma 2.1 gives

$$\sum_{k=0}^{n}\binom nk W_k3^{n-k}=W_n(0)=\sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}0^{n-3k}.$$

This yields the result.

Now we are ready to prove the following result.

Theorem 2.1. *Let p be a prime with $p>3$. Then*

$$\sum_{k=0}^{p-1}\frac{W_k}{(-3)^k}\equiv\sum_{k=0}^{p-1}\frac{W_k}{(-9)^k}$$
$$\equiv\begin{cases}-L\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=L^2+27M^2 \text{ with } L\equiv 1\pmod 3,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3\end{cases}$$

and

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-12)^k}\equiv\begin{cases}L^2\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=L^2+27M^2,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Proof. Putting $m=-9$ and $x=6$ in Lemma 2.2 we get $\sum_{k=0}^{p-1}\frac{W_k}{(-9)^k}\equiv P_{[\frac p3]}(\frac54)\pmod p$. Now applying [28, Theorem 3.2] gives the congruence for $\sum_{k=0}^{p-1}\frac{W_k}{(-9)^k}\pmod p$. Since

$\binom{p-1}{k} \equiv (-1)^k \pmod p$ for $k = 0, 1, \ldots, p-1$, using Lemma 2.3 and [5, Theorem 9.2.1] we see that

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{W_k}{(-3)^k} &\equiv \sum_{k=0}^{p-1} \binom{p-1}{k} W_k 3^{p-1-k} \\ &= \begin{cases} \binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}} \binom{p-1}{\frac{p-1}{3}} \equiv \binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}} \equiv -L \pmod p \\ \qquad\qquad \text{if } 3 \mid p-1 \text{ and } 4p = L^2 + 27M^2 \text{ with } L \equiv 1 \pmod 3, \\ 0 \pmod p \quad \text{if } 3 \mid p-2. \end{cases}\end{aligned}$$

Note that $p \mid \binom{2k}{k}$ for $k = \frac{p+1}{2}, \ldots, p-1$ and $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = (-4)^{-k}\binom{2k}{k} \pmod p$ for $k = 0, 1, \ldots, \frac{p-1}{2}$. Using Lemma 2.3 we see that

$$\begin{aligned}\sum_{k=0}^{p-1} \binom{2k}{k} \frac{W_k}{(-12)^k} &\equiv \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} \frac{W_k}{3^k} \equiv \Big(\frac{3}{p}\Big) \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} W_k \cdot 3^{\frac{p-1}{2}-k} \\ &= \begin{cases} (\frac{3}{p}) \binom{\frac{p-1}{3}}{\frac{p-1}{6}} \binom{\frac{p-1}{2}}{\frac{p-1}{6}} \equiv 2^{\frac{p-1}{3}} \binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2 \pmod p & \text{if } 3 \mid p-1, \\ 0 \pmod p & \text{if } 3 \mid p-2. \end{cases}\end{aligned}$$

Now assume $p \equiv 1 \pmod 3$. Then $p = A^2 + 3B^2$ and $4p = L^2 + 27M^2$ with $A, B, L, M \in \mathbb{Z}$ and $A \equiv L \equiv 1 \pmod 3$. By [5, p.201], $\binom{\frac{p-1}{2}}{\frac{p-1}{6}} \equiv 2A \pmod p$. If 2 is a cubic residue of p, then $2^{\frac{p-1}{3}} \equiv 1 \pmod p$. It is well known that $3 \mid B$ and so $L = -2A$. Hence

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{W_k}{(-12)^k} \equiv 2^{\frac{p-1}{3}} \binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2 \equiv (2A)^2 \equiv L^2 \pmod p.$$

Now assume that 2 is a cubic nonresidue of p. Then $2^{\frac{p-1}{3}} \not\equiv 1 \pmod p$, $3 \nmid B$ and $2 \nmid LM$. We choose the sign of M so that $M \equiv L \pmod 4$ and the sign of B so that $B \equiv A \equiv 1 \pmod 3$. From [23, p.227] we know that

$$2^{\frac{p-1}{3}} \equiv \frac{-1 - A/B}{2} \pmod p, \quad A = \frac{L - 9M}{4} \quad \text{and} \quad B = \frac{L + 3M}{4}.$$

Hence

$$\begin{aligned}\sum_{k=0}^{p-1} \binom{2k}{k} \frac{W_k}{(-12)^k} &\equiv 2^{\frac{p-1}{3}} \binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2 \equiv \frac{-1 - \frac{A}{B}}{2} \cdot 4A^2 \equiv \frac{-1 - \frac{A}{B}}{2} \cdot 4(-3B^2) \\ &= 6(A+B)B = 6\Big(\frac{L - 9M}{4} + \frac{L + 3M}{4}\Big)\frac{L + 3M}{4} \\ &= \frac{1}{4}(3L^2 - 27M^2) \equiv \frac{1}{4}(3L^2 + L^2) = L^2 \pmod p.\end{aligned}$$

This proves the remaining part and the proof is now complete.

Remark 2.1 In [37, Conjecture 1.4] Zhi-Wei Sun conjectured that if p is a prime such that $p\equiv 1\pmod 6$ and so $4p=L^2+27M^2$ with $L\equiv 1\pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{W_k}{(-9)^k}\equiv\sum_{k=0}^{p-1}\frac{W_k}{(-3)^k}\equiv -L+\frac{p}{L}\pmod{p^2};$$

if p is a prime with $p\equiv 5\pmod 6$, then $\sum_{k=0}^{p-1}\frac{W_k}{(-9)^k}\equiv 0\pmod{p^2}$.

Lemma 2.4. *Let p be an odd prime, $n,x\in\mathbb{Z}_p$ and $n(n+4x)\not\equiv 0\pmod p$. Then*

$$\Big(\frac{n+4x}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k(x)}{(n+4x)^k}\equiv\Big(\frac{-1}{p}\Big)W_{\frac{p-1}{2}}\Big(-\frac{n}{4}\Big)\equiv\Big(\frac{n}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}}\pmod p.$$

Proof. As $\binom{\frac{p-1}{2}}{k}\equiv\binom{-\frac12}{k}=\binom{2k}{k}(-4)^{-k}\pmod p$ and $p\mid\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$ for $\frac{p}{6}<k<p$, using Lemma 2.1 we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k(x)}{(n+4x)^k}\\
&\equiv\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}W_k(x)\Big(\frac{-4}{n+4x}\Big)^k\equiv\Big(\frac{-4(n+4x)}{p}\Big)\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}W_k(x)\Big(\frac{n+4x}{-4}\Big)^{\frac{p-1}{2}-k}\\
&=\Big(\frac{-n-4x}{p}\Big)W_{\frac{p-1}{2}}\Big(-\frac{n}{4}\Big)=\Big(\frac{-n-4x}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{\frac{p-1}{2}}{3k}\Big(-\frac{n}{4}\Big)^{\frac{p-1}{2}-3k}\\
&\equiv\Big(\frac{n(n+4x)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\frac{1}{(-4)^{3k}\cdot(-n/4)^{3k}}\\
&\equiv\Big(\frac{n(n+4x)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}}\pmod p.\end{aligned}$$

This proves the theorem.

Theorem 2.2. *Let p be an odd prime. Then*

$$W_{\frac{p-1}{2}}\equiv\begin{cases}4x^2\pmod p & \text{if } p\equiv 1\pmod 4 \text{ and so } p=x^2+y^2 \text{ with } 2\nmid x,\\ 0\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Proof. Since $W_1=-3$ we see that the result is true for $p=3$. Now assume $p>3$. Putting $n=12$ in Lemma 2.4 yields

$$W_{\frac{p-1}{2}}=W_{\frac{p-1}{2}}(-3)\equiv\Big(\frac{-12}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\pmod p.$$

In [19] Mortenson proved the congruence

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\begin{cases}(\frac{-3}{p})4x^2\pmod p & \text{if } 4\mid p-1 \text{ and so } p=x^2+y^2 \text{ with } 2\nmid x,\\ 0\pmod p & \text{if } 4\mid p-3,\end{cases}$$

which was conjectured by Rodriguez-Villegas in 2003. Now combining the above gives the result.

Theorem 2.3. *Let p be a prime with $p>3$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{54^k}\equiv\begin{cases}(\frac{p}{3})(4x^2-2p)\pmod p & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\ 0\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Proof. It is easy to check the result for $p=11$. Now assume that $p\neq 11$. Taking $n=66$ and $x=-3$ in Lemma 2.4 and then applying [29, Theorem 4.3] deduces the result.

Theorem 2.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{8^k}\equiv\begin{cases}4x^2-2p\pmod p & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ 0\pmod p & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Proof. It is easy to check the result for $p=5$. Now assume that $p>5$. Taking $n=20$ and $x=-3$ in Lemma 2.4 and then applying [29, Theorem 4.4] we deduce the result.

Theorem 2.5. *Let p be a prime with $p\neq 2,3,7$. Then*

$$\Big(\frac{-3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-27)^k}\equiv\Big(\frac{-3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{243^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod p & \text{if } p=x^2+7y^2\equiv 1,2,4\pmod 7,\\ 0\pmod p & \text{if } p\equiv 3,5,6\pmod 7.\end{cases}$$

Proof. It is easy to check the result for $p=5,17$. Now assume that $p\neq 5,17$. Taking $n=-15$ and $x=-3$ in Lemma 2.4 and then applying [29, Theorem 4.7] we deduce the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-27)^k}\pmod p$. Taking $n=255$ and $x=-3$ in Lemma 2.4 and then applying [29, Theorem 4.7] we deduce the remaining part.

Theorem 2.6. *Let p be a prime and $p\neq 2,11$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-44)^k}\equiv\begin{cases}x^2-2p\pmod p & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=x^2+11y^2,\\ 0\pmod p & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Proof. It is easy to check the result for $p=3$. Now assume that $p\neq 3$. Taking $n=-32$ and $x=-3$ in Lemma 2.4 and then applying [29, Theorem 4.8] we deduce the result.

Theorem 2.7. *Let p be a prime with $p\neq 2,3,19$. Then*

$$\Big(\frac{-3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-108)^k}\equiv\begin{cases}x^2-2p\pmod p & \text{if } (\frac{p}{19})=1 \text{ and so } 4p=x^2+19y^2,\\ 0\pmod p & \text{if } (\frac{p}{19})=-1.\end{cases}$$

Proof. Taking $n=-96$ and $x=-3$ in Lemma 2.4 and then applying [29, Theorem 4.9] we deduce the result.

Using Lemma 2.4 and [29, Theorem 4.9] one can also deduce the following results.

Theorem 2.8. *Let p be a prime, $p \neq 2, 3, 43$. Then*

$$\Big(\frac{-3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-972)^k} \equiv \begin{cases} x^2-2p \pmod p & \text{if } (\frac{p}{43})=1 \text{ and so } 4p=x^2+43y^2,\\ 0 \pmod p & \text{if } (\frac{p}{43})=-1.\end{cases}$$

Theorem 2.9. *Let p be a prime with $p \neq 2, 3, 7, 67$. Then*

$$\Big(\frac{-3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-5292)^k} \equiv \begin{cases} x^2-2p \pmod p & \text{if } (\frac{p}{67})=1 \text{ and so } 4p=x^2+67y^2,\\ 0 \pmod p & \text{if } (\frac{p}{67})=-1.\end{cases}$$

Theorem 2.10. *Let p be a prime with $p \neq 2, 3, 7, 11, 163$. Then*

$$\begin{aligned}&\Big(\frac{-3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-640332)^k}\\ &\equiv \begin{cases} x^2-2p \pmod p & \text{if } (\frac{p}{163})=1 \text{ and so } 4p=x^2+163y^2,\\ 0 \pmod p & \text{if } (\frac{p}{163})=-1.\end{cases}\end{aligned}$$

Theorem 2.11. *Suppose that $p>3$ is a prime, $x \in \mathbb{Z}_p$ and $x(x^3+27)(x^3-216)(x^2+6x-18) \not\equiv 0 \pmod p$. Then*

$$\begin{aligned}W_{p-1}(x)^2 &\equiv \Big(\sum_{k=0}^{p-1}\frac{W_k}{(-x-3)^k}\Big)^2 \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(-\frac{x^3+27}{x^6}\Big)^k\\ &\equiv \Big(\frac{x(x^3-216)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(-\frac{x^3+27}{x(x^3-216)}\Big)^{3k}\\ &\equiv \Big(\frac{x^3+27}{p}\Big)W_{\frac{p-1}{2}}\Big(\frac{x(x^3-216)}{4(x^3+27)}\Big)\\ &\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\Big(-\frac{x^3+27}{(x^2+6x-18)^2}\Big)^k W_k \pmod p.\end{aligned}$$

Proof. From Lemma 2.2 we see that $W_{p-1}(x) \equiv \sum_{k=0}^{p-1}\frac{W_k}{(-x-3)^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-x)^{3k}} \pmod p$. By [28, Theorem 2.1],

$$\Big(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}m^k\Big)^2 \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}(m(1-27m))^k \pmod{p^2}.$$

Hence,

$$\begin{aligned}&W_{p-1}(x)^2\\ &\equiv \Big(\sum_{k=0}^{p-1}\frac{W_k}{(-x-3)^k}\Big)^2 \equiv \Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-x)^{3k}}\Big)^2 \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(-\frac{1}{x^3}\Big(1+\frac{27}{x^3}\Big)\Big)^k \pmod p.\end{aligned}$$

By [30, Theorem 2.2], for $t \in \mathbb{Z}_p$ with $4t \not\equiv \pm 5 \pmod p$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{1-t^2}{108}\Big)^k \equiv \Big(\frac{4t+5}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{(t+1)(1-t)^3}{432(4t+5)^3}\Big)^k \pmod p.$$

Taking $t = -1 - \frac{54}{x^3}$ gives

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(-\frac{1}{x^3}\Big(1+\frac{27}{x^3}\Big)\Big)^k\\ &\equiv \Big(\frac{x(x^3-216)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(-\frac{x^3+27}{x(x^3-216)}\Big)^{3k} \pmod p.\end{aligned}$$

Set $n = -\frac{x(x^3-216)}{x^3+27}$. Then $n - 12 = -\frac{x^4+12x^3-216x+324}{x^3+27} = -\frac{(x^2+6x-18)^2}{x^3+27}$. From Lemma 2.4 (with $x=-3$) and the fact $\binom{(p-1)/2}{r} \equiv 4^{-r}\binom{2r}{r} \pmod p$ for $0 \le r \le \frac{p-1}{2}$ we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\Big(-\frac{x^3+27}{(x^2+6x-18)^2}\Big)^k W_k\\ &\equiv \Big(\frac{n-12}{p}\Big)(-1)^{\frac{p-1}{2}}W_{\frac{p-1}{2}}\Big(-\frac{n}{4}\Big) = \Big(\frac{x^3+27}{p}\Big)W_{\frac{p-1}{2}}\Big(\frac{x(x^3-216)}{4(x^3+27)}\Big)\\ &= \Big(\frac{x^3+27}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{\frac{p-1}{2}}{3k}\Big(\frac{x(x^3-216)}{4(x^3+27)}\Big)^{\frac{p-1}{2}-3k}\\ &\equiv \Big(\frac{x(x^3-216)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(-\frac{x^3+27}{x(x^3-216)}\Big)^{3k} \pmod p.\end{aligned}$$

Now putting all the above together proves the theorem.

Corollary 2.1. *Suppose that $p > 3$ is a prime, $x \in \mathbb{Z}_p$ and $(x+3)(1+9x+27x^2)(1+9x)(1+27x^2)(1-27x^2) \not\equiv 0 \pmod p$. Then*

$$\Big(\sum_{k=0}^{p-1}W_kx^k\Big)^2 \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x(1+9x+27x^2)}{(1-27x^2)^2}\Big)^k W_k \pmod p.$$

Proof. Substituting x with $-\frac{1}{x} - 3$ in Theorem 2.11 yields

$$\begin{aligned}\Big(\sum_{k=0}^{p-1}W_kx^k\Big)^2 &\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\Big(-\frac{-(\frac{1}{x}+3)^3+27}{(\frac{1}{x^2}-27)^2}\Big)^k W_k\\ &= \sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x(1+9x+27x^2)}{(1-27x^2)^2}\Big)^k W_k \pmod p.\end{aligned}$$

This proves the corollary.

We remark that Corollary 2.1 is the p-analogue of (1.3).

3. Congruences for generalized Apéry-like numbers

In 2010, Jarvis and Verrill [13] established a relation between u_n and u_{p-1-n} modulo p for $\{A'_n\},\{a_n\},\{f_n\}$ and $\{S_n\}$, where p is an odd prime. Inspired by (1.1) and (1.2), we introduce generalized Apéry-like numbers and prove a vast generalization of those congruences given in [13].

Theorem 3.1. *Suppose $r\in\mathbb{Z}^+$ and $c\in\mathbb{Z}$ with $c\neq 0$. Let $b(n)$ be the polynomial of n with integral coefficients and the property $b(-1-n)=(-1)^r b(n)$ for any $n\in\mathbb{Z}$. Define the sequence $\{u_n\}$ by*

$$u_0=1,\ u_1=b(0)\quad\text{and}\quad (n+1)^r u_{n+1}=b(n)u_n-cn^r u_{n-1}\ (n\ge 1). \tag{3.1}$$

Suppose that p is an odd prime with $p\nmid c$ and $u_p\in\mathbb{Z}_p$. For $n=0,1,2,\dots,p-1$ we have

$$u_n\equiv u_{p-1}c^n u_{p-1-n}\equiv\begin{cases}\left(\frac{c}{p}\right)c^n u_{p-1-n}\pmod p & \text{if } p\nmid u_{\frac{p-1}{2}},\\ (-1)^{r-1}\left(\frac{c}{p}\right)c^n u_{p-1-n}\pmod p & \text{if } p\mid u_{\frac{p-1}{2}}.\end{cases}$$

In particular,

$$u_{p-1}\equiv\begin{cases}\left(\frac{c}{p}\right)\pmod p & \text{if } p\nmid u_{\frac{p-1}{2}},\\ (-1)^{r-1}\left(\frac{c}{p}\right)\pmod p & \text{if } p\mid u_{\frac{p-1}{2}}.\end{cases}$$

Proof. By (3.1), for $n\in\{0,1,\dots,p-1\}$, $u_n\in\mathbb{Z}_p$, $(p-n)^r u_{p-n}=b(p-1-n)u_{p-1-n}-c(p-1-n)^r u_{p-2-n}$ and so $(-n)^r u_{p-n}\equiv b(-1-n)u_{p-1-n}-c(-n-1)^r u_{p-2-n}\pmod p$. Since $b(-1-n)=(-1)^r b(n)$ we get $n^r u_{p-n}\equiv b(n)u_{p-1-n}-c(n+1)^r u_{p-2-n}\pmod p$. Multiplying c^n on both sides gives

$$(n+1)^r c^{n+1}u_{p-2-n}\equiv b(n)c^n u_{p-1-n}-cn^r\cdot c^{n-1}u_{p-n}\pmod p. \tag{3.2}$$

By (3.1), $p^r u_p=b(p-1)u_{p-1}-c(p-1)^r u_{p-2}$. Thus $b(p-1)u_{p-1}\equiv c(-1)^r u_{p-2}\pmod p$. Since $b(p-1)\equiv b(-1)=(-1)^r b(0)\pmod p$ we see that $b(0)u_{p-1}\equiv cu_{p-2}\pmod p$. If $p\mid u_{p-1}$, we must have $p\mid u_{p-2}$ and so $p\mid u_{p-3}$ by (3.1). If $u_{p-(m-1)}\equiv u_{p-m}\equiv 0\pmod p$ for $m\in\{2,3,\dots,p-1\}$, then $u_{p-(m+1)}\equiv 0\pmod p$ by (3.1). Hence $u_{p-2}\equiv u_{p-3}\equiv\cdots\equiv u_1\equiv u_0\equiv 0\pmod p$. But $u_0=1$. This is a contradiction. Therefore $p\nmid u_{p-1}$. Set $v_n=c^n u_{p-1-n}/u_{p-1}$. Then $v_0=1=u_0$ and $v_1=cu_{p-2}/u_{p-1}\equiv b(0)=u_1\pmod p$. By (3.2), for $n=1,2,\dots,p-1$ we have $(n+1)^r v_{n+1}\equiv b(n)v_n-cn^r v_{n-1}\pmod p$. Hence $u_n\equiv v_n=c^n u_{p-1-n}/u_{p-1}\pmod p$ for $n=0,1,\dots,p-1$ and so $u_{p-1}\equiv c^{p-1}u_0/u_{p-1}\pmod p$, which implies $u_{p-1}^2\equiv c^{p-1}\equiv 1\pmod p$ and so $u_{p-1}\equiv\varepsilon_p\pmod p$ for $\varepsilon_p\in\{1,-1\}$. This yields $u_n\equiv c^n u_{p-1-n}/u_{p-1}\equiv\varepsilon_p c^n u_{p-1-n}\pmod p$. Taking $n=\frac{p-1}{2}$ gives $u_{\frac{p-1}{2}}\equiv\varepsilon_p c^{\frac{p-1}{2}}u_{\frac{p-1}{2}}\equiv\varepsilon_p(\frac{c}{p})u_{\frac{p-1}{2}}\pmod p$. Hence, if $p\nmid u_{\frac{p-1}{2}}$, then $\varepsilon_p(\frac{c}{p})\equiv 1\pmod p$, $\varepsilon_p=(\frac{c}{p})$ and so $u_n\equiv(\frac{c}{p})c^n u_{p-1-n}\pmod p$. Now assume that $p\mid u_{\frac{p-1}{2}}$. By the above argument, $u_{\frac{p+1}{2}}\equiv\varepsilon_p c^{\frac{p+1}{2}}u_{\frac{p-3}{2}}\equiv\varepsilon_p c(\frac{c}{p})u_{\frac{p-3}{2}}\pmod p$. By (3.1),

$$\Big(\frac{p+1}{2}\Big)^r u_{\frac{p+1}{2}}=b\Big(\frac{p-1}{2}\Big)u_{\frac{p-1}{2}}-c\Big(\frac{p-1}{2}\Big)^r u_{\frac{p-3}{2}}\equiv -c\Big(\frac{p-1}{2}\Big)^r u_{\frac{p-3}{2}}\pmod p.$$

Namely, $u_{\frac{p+1}{2}} \equiv (-1)^{r-1} c u_{\frac{p-3}{2}} \pmod p$. Hence $c(\varepsilon_p(\frac{c}{p}) - (-1)^{r-1}) u_{\frac{p-3}{2}} \equiv u_{\frac{p+1}{2}} - u_{\frac{p+1}{2}} = 0 \pmod p$. If $p \mid u_{\frac{p-3}{2}}$, since $p \mid u_{\frac{p-1}{2}}$ we see that $u_{\frac{p-5}{2}} \equiv \cdots \equiv u_0 \equiv 0 \pmod p$ by (3.1). But $u_0 = 1$. Therefore $p \nmid u_{\frac{p-3}{2}}$ and so $\varepsilon_p(\frac{c}{p}) = (-1)^{r-1}$. This yields $u_n \equiv \varepsilon_p c^n u_{p-1-n} = (-1)^{r-1}(\frac{c}{p}) c^n u_{p-1-n} \pmod p$, which completes the proof.

Corollary 3.1. *Let $p > 3$ be a prime and $n \in \{0, 1, \dots, p-1\}$. Then*

$$\begin{aligned} &P_n(x) \equiv P_{p-1-n}(x) \pmod p, \quad A_n \equiv A_{p-1-n} \pmod p, \\ &D_n \equiv 64^n D_{p-1-n} \pmod p, \quad b_n \equiv 81^n b_{p-1-n} \pmod p, \\ &T_n \equiv 16^n T_{p-1-n} \pmod p, \quad W_n \equiv \Big(\frac{p}{3}\Big) 27^n W_{p-1-n} \pmod p, \\ &Q_n \equiv \Big(\frac{p}{3}\Big) 72^n Q_{p-1-n} \pmod p. \end{aligned}$$

Proof. Taking $u_n = P_n(x), A_n, D_n, b_n, T_n$ in Theorem 3.1 yields the first five congruences. Since $\binom{p-1}{m} \equiv (-1)^m \pmod p$, using [25, Corollary 2.2] we deduce that

$$W_{p-1} = \sum_{k=0}^{[p/3]} \binom{2k}{k}\binom{3k}{k}\binom{p-1}{3k}(-3)^{p-1-3k} \equiv \sum_{k=0}^{[p/3]} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \Big(\frac{p}{3}\Big) \pmod p.$$

Recall that $(n+1)^2 W_{n+1} = (-9n(n+1)-3)W_n - 27n^2 W_{n-1}$ $(n \ge 1)$. Applying Theorem 3.1 yields the result for W_n. Using [32, Lemma 2.4] and [25, Corollary 2.2] we see that

$$Q_{p-1} = \sum_{k=0}^{p-1} \binom{p-1}{k} (-8)^{p-1-k} f_k \equiv \sum_{k=0}^{p-1} \frac{f_k}{8^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \Big(\frac{p}{3}\Big) \pmod p.$$

Recall that $(n+1)^2 Q_{n+1} = (-17n(n+1)-6)Q_n - 72n^2 Q_{n-1}$ $(n \ge 1)$. Applying Theorem 3.1 yields the result for Q_n.

Theorem 3.2. *Let $\{u_n\}$ be given in Theorem 3.1, and let p be an odd prime. Suppose that $u_m \in \mathbb{Z}_p$ for $m = 0, 1, 2, \dots$ and $k \in \mathbb{Z}^+$. Then*

$$u_{kp+n} \equiv u_{kp} u_n \pmod p \quad \textit{for} \quad n = 0, 1, \dots, p-1.$$

Proof. We prove the theorem by induction on n. Clearly the result is true for $n = 0$ since $u_0 = 1$. By (3.1), $(kp+1)^r u_{kp+1} = b(kp)u_{kp} - c(kp)^r u_{kp-1}$. Thus, $u_{kp+1} \equiv b(kp)u_{kp} \equiv b(0)u_{kp} = u_1 u_{kp} \pmod p$. This shows that the result is also true for $n = 1$. Now assume $2 \le m \le p-1$ and the result holds for $n < m$. From (3.1) and the inductive hypothesis we see that

$$\begin{aligned} m^r u_{kp+m} &\equiv b(kp+m-1)u_{kp+m-1} - c(kp+m-1)^r u_{kp+m-2} \\ &\equiv b(m-1)u_{kp+m-1} - c(m-1)^r u_{kp+m-2} \\ &\equiv b(m-1)u_{kp}u_{m-1} - c(m-1)^r u_{kp}u_{m-2} = u_{kp} \cdot m^r u_m \pmod p. \end{aligned}$$

Since $p \nmid m$ we get $u_{kp+m} \equiv u_{kp} u_m \pmod p$. This shows that the result is true for $n = m$. Hence the theorem is proved by induction.

Corollary 3.2. *Let $\{u_n\}$ be given in Theorem 3.1, and let p be an odd prime. Suppose $u_m \in \mathbb{Z}_p$ and $u_{mp} \equiv u_m \pmod p$ for $m = 1, 2, 3, \ldots$. For $n \in \mathbb{Z}^+$ write $n = n_0 + n_1 p + \cdots + n_s p^s$, where $n_0, n_1, \ldots, n_s \in \{0, 1, \ldots, p-1\}$. Then we have the Lucas congruence $u_n \equiv u_{n_0} u_{n_1} \cdots u_{n_s} \pmod p$.*

Proof. Set $k = n_1 + n_2 p + \cdots + n_s p^{s-1}$. Then $n = kp + n_0$. By Theorem 3.2,

$$\begin{aligned} u_n &\equiv u_{kp} u_{n_0} \equiv u_k u_{n_0} = u_{n_1+n_2p+\cdots+n_sp^{s-1}} u_{n_0} \equiv u_{n_0} u_{n_1} u_{(n_2+n_3p+\cdots+n_sp^{s-2})p} \\ &\equiv u_{n_0} u_{n_1} u_{n_2+n_3p+\cdots+n_sp^{s-2}} \equiv u_{n_0} u_{n_1} u_{n_2} u_{n_3+n_4p+\cdots+n_sp^{s-3}} \\ &\equiv \cdots \equiv u_{n_0} u_{n_1} \cdots u_{n_{s-2}} u_{n_{s-1}+n_sp} \equiv u_{n_0} u_{n_1} \cdots u_{n_{s-1}} u_{n_s} \pmod p. \end{aligned}$$

This proves the corollary.

Remark 3.1 Suppose that p is a prime. For $m, n \in \mathbb{Z}^+$ with $m \le n$ write $n = n_0 + n_1 p + \cdots + n_s p^s$ and $m = m_0 + m_1 p + \cdots + m_s p^s$, where $n_0, \ldots, n_s, m_0, \ldots, m_s \in \{0, 1, \ldots, p-1\}$. Then $\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_s}{m_s} \pmod p$. This is called Lucas theorem. From [13] and [17] we know that many Apéry-like numbers satisfy the Lucas congruences.

Theorem 3.3. *Let $\{u_n\}$ be given by (3.1). Then*

$$\sum_{k=0}^{n-1} b(k)(-c)^{n-1-k} u_k^2 = n^r u_n u_{n-1} \quad (n = 1, 2, 3, \ldots).$$

Thus, if p is an odd prime, $p \nmid c$ and $u_m \in \mathbb{Z}_p$ for $m \in \mathbb{Z}^+$, then

$$\sum_{k=0}^{p-1} \frac{b(k)}{(-c)^k} u_k^2 \equiv 0 \pmod{p^r}.$$

Proof. Since

$$\frac{(k+1)^r u_{k+1}}{(-c)^k} - \frac{k^r u_{k-1}}{(-c)^{k-1}} = \frac{(k+1)^r u_{k+1} + c k^r u_{k-1}}{(-c)^k} = \frac{b(k) u_k}{(-c)^k},$$

we see that

$$\sum_{k=0}^{n-1} \frac{b(k)}{(-c)^k} u_k^2 = \sum_{k=0}^{n-1} \Big(\frac{(k+1)^r u_{k+1} u_k}{(-c)^k} - \frac{k^r u_k u_{k-1}}{(-c)^{k-1}} \Big) = \frac{n^r u_n u_{n-1}}{(-c)^{n-1}}.$$

This yields the result.

As an example, taking $u_n = P_n(x)$ in Theorem 3.3 gives

$$\sum_{k=0}^{n-1} (-1)^{n-1-k} (2k+1) P_k(x)^2 = n \frac{P_n(x) P_{n-1}(x)}{x}. \tag{3.3}$$

4. Conjectures on congruences involving binomial coefficients and Apéry-like numbers

The Bernoulli numbers $\{B_n\}$, Euler numbers $\{E_n\}$ and the sequence $\{U_n\}$ are defined by

$$\begin{aligned} &B_0 = 1, \quad \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 \quad (n \ge 2),\\ &E_0 = 1, \quad E_n = -\sum_{k=1}^{[n/2]}\binom{n}{2k}E_{n-2k} \quad (n \ge 1),\\ &U_0 = 1, \quad U_n = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k} \quad (n \ge 1). \end{aligned}$$

For congruences involving B_n, E_n and U_n see [22,24,26].

Based on calculations with Maple, we pose the following challenging conjectures:

Conjecture 4.1. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} &A_{p-1} \equiv 1 + \frac{2}{3}p^3B_{p-3} \pmod{p^4}, \quad D_{p-1} \equiv 64^{p-1} - \frac{p^3}{6}B_{p-3} \pmod{p^4},\\ &b_{p-1} \equiv 81^{p-1} - \frac{2}{27}p^3B_{p-3} \pmod{p^4}, \quad T_{p-1} \equiv 16^{p-1} + \frac{p^3}{4}B_{p-3} \pmod{p^4}. \end{aligned}$$

Remark 4.1 In [33] the author proved that $T_{p-1} \equiv 16^{p-1} \pmod{p^3}$ for any prime $p > 3$.

Conjecture 4.2. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} &A'_{p-1} \equiv 1 + \frac{5}{3}p^3B_{p-3} \pmod{p^4}, \ f_{p-1} \equiv 8^{p-1} + \frac{5}{8}p^3B_{p-3} \pmod{p^4},\\ &S_{p-1} \equiv (-1)^{\frac{p-1}{2}}32^{p-1} + p^2E_{p-3} \pmod{p^3}, \ a_{p-1} \equiv \left(\frac{p}{3}\right)9^{p-1} + p^2U_{p-3} \pmod{p^3},\\ &W_{p-1} \equiv \left(\frac{p}{3}\right)27^{p-1} + p^2U_{p-3} \pmod{p^3}, \ Q_{p-1} \equiv \left(\frac{p}{3}\right)72^{p-1} + \frac{5}{2}p^2U_{p-3} \pmod{p^3}. \end{aligned}$$

Remark 4.2 Let $p > 3$ be a prime. In [38] Z.W. Sun proved a congruence equivalent to $f_{p-1} \equiv 8^{p-1} \pmod{p^3}$. In [33] the author proved a congruence equivalent to $S_{p-1} \equiv (-1)^{\frac{p-1}{2}}32^{p-1} \pmod{p^2}$.

Let p be an odd prime. In 2000 Ahlgren and Ono[2] proved Beukers' conjecture $A_{\frac{p-1}{2}} \equiv c(p) \pmod{p^2}$, where $\{c(n)\}$ is given by

$$\Phi_1(q) = q\prod_{k=1}^{\infty}(1-q^{2k})^4(1-q^{4k})^4 = \sum_{n=1}^{\infty}c(n)q^n \quad (|q| < 1).$$

It is well known that $\Phi_1(\mathrm{e}^{2\pi iz})$ is a newform in $S_4(\Gamma_0(8))$. For $|q| < 1$ define

$$\Phi_4(q) = q\prod_{k=1}^{\infty}(1-q^{4k})^6 = \sum_{n=1}^{\infty}\alpha(n)q^n,$$

$$\Phi_2(q) = q\prod_{k=1}^{\infty}(1-q^k)^2(1-q^{2k})(1-q^{4k})(1-q^{8k})^2 = \sum_{n=1}^{\infty}\beta(n)q^n,$$
$$\Phi_3(q) = q\prod_{k=1}^{\infty}(1-q^{2k})^3(1-q^{6k})^3 = \sum_{n=1}^{\infty}\gamma(n)q^n.$$

It is known that $\Phi_4(\mathrm{e}^{2\pi iz})$ is a weight 3 newform with complex multiplication by $\mathbb{Q}(\sqrt{-1})$, and for $m \in \{2,3\}$ $\Phi_m(\mathrm{e}^{2\pi iz})$ is a weight 3 newform with complex multiplication by $\mathbb{Q}(\sqrt{-m})$. More precisely,

$$\Phi_4(\mathrm{e}^{2\pi iz}) \in S_3\Big(\Gamma_0(16), (\frac{-4}{\cdot})\Big),$$
$$\Phi_3(\mathrm{e}^{2\pi iz}) \in S_3\Big(\Gamma_0(12), (\frac{-3}{\cdot})\Big), \quad \Phi_2(\mathrm{e}^{2\pi iz}) \in S_3\Big(\Gamma_0(8), (\frac{-8}{\cdot})\Big),$$

where $(\frac{a}{\cdot})$ is the Legendre-Jacobi-Kronecker symbol. See [11,18,21]. From [21, (14.2)] we know that for odd prime p,

$$\alpha(p) = \begin{cases} 4x^2-2p & \text{if } p = x^2+4y^2 \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{4.1}$$

$$\beta(p) = \begin{cases} 4x^2-2p & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ 0 & \text{if } p \equiv 5,7 \pmod 8, \end{cases} \tag{4.2}$$

$$\gamma(p) = \begin{cases} 4x^2-2p & \text{if } p = x^2+3y^2 \equiv 1 \pmod 3, \\ 0 & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{4.3}$$

In [1] Ahlgren proved Beukers' conjecture:

$$A'_{\frac{p-1}{2}} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 4 \text{ and so } p = x^2+4y^2, \\ 0 \pmod{p^2} & \text{if } p > 3 \text{ and } p \equiv 3 \pmod 4. \end{cases}$$

This is equivalent to $A'_{\frac{p-1}{2}} \equiv \alpha(p) \pmod{p^2}$ for $p > 3$. Using modular forms with complex multiplication, Gomez, McCarthy and Young [11] proved that for prime $p > 2$ and $r \in \mathbb{Z}^+$,

$$A'_{\frac{p^r-1}{2}} \equiv \begin{cases} (x+yi)^{2r}+(x-yi)^{2r} \pmod{p^2} & \text{if } p = x^2+y^2 \equiv 1 \pmod 4 \text{ with } 2\nmid x, \\ 0 \pmod{p^2} & \text{if } p > 3 \text{ and } p \equiv 3 \pmod 4, \end{cases}$$
$$a_{\frac{p^r-1}{2}} \equiv \begin{cases} (x+y\sqrt{-3})^{2r}+(x-y\sqrt{-3})^{2r} \pmod p & \text{if } p = x^2+3y^2 \equiv 1 \pmod 6, \\ 0 \pmod p & \text{if } p \equiv 5 \pmod 6, \end{cases}$$
$$(-1)^{\frac{p^r-1}{2}} f_{\frac{p^r-1}{2}}$$
$$\equiv \begin{cases} (x+y\sqrt{-2})^{2r}+(x-y\sqrt{-2})^{2r} \pmod p & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

By [21, (13.1)], for odd prime p, $m \in \{1,3,5,\dots\}$ and $r \in \{2,3,4,\dots\}$,

$$A'_{\frac{mp^r-1}{2}} \equiv \begin{cases} (4x^2-2p)A'_{\frac{mp^{r-1}-1}{2}} - p^2 A'_{\frac{mp^{r-2}-1}{2}} \pmod{p^r} & \text{if } p = x^2+4y^2 \equiv 1 \pmod 4, \\ p^2 A'_{\frac{mp^{r-2}-1}{2}} \pmod{p^r} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

$$f_{\frac{mp^r-1}{2}} \equiv \begin{cases} (-1)^{\frac{p-1}{2}}(4x^2-2p)f_{\frac{mp^{r-1}-1}{2}} - p^2 f_{\frac{mp^{r-2}-1}{2}} \pmod{p^r} & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ p^2 f_{\frac{mp^{r-2}-1}{2}} \pmod{p^r} & \text{if } p \equiv 5,7 \pmod 8, \end{cases}$$

$$a_{\frac{mp^r-1}{2}} \equiv \begin{cases} (4x^2-2p)a_{\frac{mp^{r-1}-1}{2}} - p^2 a_{\frac{mp^{r-2}-1}{2}} \pmod{p^r} & \text{if } p = x^2+3y^2 \equiv 1 \pmod 3, \\ p^2 a_{\frac{mp^{r-2}-1}{2}} \pmod{p^r} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Now we present several conjectures, which can be viewed as generalizations of the above results.

Conjecture 4.3. *Let p be a prime of the form $4k+3$. Then*

$$3\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^2 A'_{\frac{p-1}{2}} \equiv p^2 \pmod{p^3},$$
$$A'_{\frac{mp^r-1}{2}} \equiv p^2 A'_{\frac{mp^{r-2}-1}{2}} \pmod{p^{2r}} \quad \text{for } m \in \{1,3,5,\dots\} \text{ and } r \in \{2,3,4,\dots\}.$$

Conjecture 4.4. *Let p be an odd prime, $m \in \{1,3,5,\dots\}$ and $r \in \{2,3,4,\dots\}$. Then*

$$a_{\frac{mp^r-1}{2}} \equiv p^2 a_{\frac{mp^{r-2}-1}{2}} \pmod{p^{2r-1}} \quad \text{for } p \equiv 5 \pmod 6,$$
$$f_{\frac{mp^r-1}{2}} \equiv p^2 f_{\frac{mp^{r-2}-1}{2}} \pmod{p^{2r-1}} \quad \text{for } p \equiv 5,7 \pmod 8,$$
$$S_{\frac{mp^r-1}{2}} \equiv p^2 S_{\frac{mp^{r-2}-1}{2}} \pmod{p^{2r-1}} \quad \text{for } p \equiv 5,7 \pmod 8,$$
$$W_{\frac{mp^r-1}{2}} \equiv p^2 W_{\frac{mp^{r-2}-1}{2}} \pmod{p^{2r-1}} \quad \text{for } p \equiv 3 \pmod 4,$$
$$Q_{\frac{mp^r-1}{2}} \equiv p^2 Q_{\frac{mp^{r-2}-1}{2}} \pmod{p^{2r-1}} \quad \text{for } p \equiv 13,17,19,23 \pmod{24}.$$

Conjecture 4.5. *Suppose that p is an odd prime, $m \in \{1,3,5,\dots\}$ and $r \in \{2,3,4,\dots\}$.*
(i) *If $p \equiv 1,3 \pmod 8$ and so $p = x^2+2y^2$, then*

$$S_{\frac{mp^r-1}{2}} \equiv (4x^2-2p)S_{\frac{mp^{r-1}-1}{2}} - p^2 S_{\frac{mp^{r-2}-1}{2}} \pmod{p^r}.$$

(ii) *If $p \equiv 1 \pmod 4$ and so $p = x^2+4y^2$, then*

$$W_{\frac{mp^r-1}{2}} \equiv (4x^2-2p)W_{\frac{mp^{r-1}-1}{2}} - p^2 W_{\frac{mp^{r-2}-1}{2}} \pmod{p^r}.$$

(iii) *If $p \equiv 1,7 \pmod{24}$ and so $p = x^2+6y^2$, then*

$$Q_{\frac{mp^r-1}{2}} \equiv (-1)^{\frac{p-1}{2}}(4x^2-2p)Q_{\frac{mp^{r-1}-1}{2}} - p^2 Q_{\frac{mp^{r-2}-1}{2}} \pmod{p^r}.$$

(iv) *If $p \equiv 5,11 \pmod{24}$ and so $p = 2x^2+3y^2$, then*

$$Q_{\frac{mp^r-1}{2}} \equiv (-1)^{\frac{p+1}{2}}(8x^2-2p)Q_{\frac{mp^{r-1}-1}{2}} - p^2 Q_{\frac{mp^{r-2}-1}{2}} \pmod{p^r}.$$

Conjecture 4.6. *Let $\{u_n\}$ be one of the six sequences $\{A'_n\}$, $\{f_n\}$, $\{S_n\}$, $\{a_n\}$, $\{Q_n\}$ and $\{W_n\}$, and $c=-1,-8,32,9,72$ or 27 according as $u_n=A'_n,f_n,S_n,a_n,Q_n$ or W_n. Suppose that p is an odd prime with $p\nmid c$. Then*

$$4u_{\frac{mp^2-1}{2}}\equiv(5-c^{p-1})u_{\frac{p-1}{2}}u_{\frac{mp-1}{2}}\pmod{p^2}\quad\text{for}\quad m=1,3,5,\ldots.$$

Comparing Conjecture 4.6 with the case $r=2$ in [21,(13.1)] and Conjecture 4.5 suggests the following conjecture:

Conjecture 4.7. *Suppose that p is an odd prime.*

(i) *If $p\equiv 1\pmod 3$ and so $p=x^2+3y^2$, then $a_{\frac{p-1}{2}}\equiv(9^{p-1}+3)x^2-2p\pmod{p^2}$.*

(ii) *If $p\equiv 1\pmod 4$ and so $p=x^2+4y^2$, then $W_{\frac{p-1}{2}}\equiv(27^{p-1}+3)x^2-2p\pmod{p^2}$.*

(iii) *If $p\equiv 1,7\pmod{24}$ and so $p=x^2+6y^2$, then $(\frac{3}{p})Q_{\frac{p-1}{2}}\equiv(72^{p-1}+3)x^2-2p\pmod{p^2}$; if $p\equiv 5,11\pmod{24}$ and so $p=2x^2+3y^2$, then $(\frac{3}{p})Q_{\frac{p-1}{2}}\equiv(72^{p-1}+3)\cdot 2x^2-2p\pmod{p^2}$.*

Remark 4.3 Let p be an odd prime. By (4.3), Conjecture 4.7(i) is equivalent to

$$a_{\frac{p-1}{2}}\equiv\Big(1+\frac14(9^{p-1}-1)\Big)\gamma(p)\pmod{p^2}\quad\text{for}\quad p\equiv 1\pmod 3.$$

By (4.1), Conjecture 4.7(ii) is equivalent to

$$W_{\frac{p-1}{2}}\equiv\Big(1+\frac14(27^{p-1}-1)\Big)\alpha(p)\pmod{p^2}\quad\text{for}\quad p\equiv 1\pmod 4.$$

By (4.2), the first congruence in [32, Conjecture 3.1] is equivalent to

$$f_{\frac{p-1}{2}}\equiv(-1)^{\frac{p-1}{2}}\Big(1+\frac14(8^{p-1}-1)\Big)\beta(p)\pmod{p^2}\quad\text{for}\quad p\equiv 1,3\pmod 8,$$

and the first part of [31, Conjecture 2.2] is equivalent to

$$S_{\frac{p-1}{2}}\equiv\Big(1+\frac14(32^{p-1}-1)\Big)\beta(p)\pmod{p^2}\quad\text{for}\quad p\equiv 1,3\pmod 8.$$

Let $p>3$ ba a prime. In [34,35] Z.W. Sun conjectured congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k\pmod{p^2}$ with $m=1,-8,16,-64,256,-512,4096$. Such conjectures were proved by the author in [27], and later proved by Kibelbek et al in [15]. In [35] Z.W. Sun also conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv(-1)^{\frac{p-1}{4}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k}\pmod{p^3}\quad\text{for}\quad p\equiv 1\pmod 4,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\Big(\frac{-1}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k}\pmod{p^3}\quad\text{for}\quad p\equiv 1,2,4\pmod 7,$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\Big(\frac{-1}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}\pmod{p^3}\quad\text{for}\quad p\equiv 1\pmod 3.$$

In light of related calculations on Maple we made the following conjectures.

Conjecture 4.8. *Let p be a prime with $p\ne 2,3,7$.*

(i) *If* $p\equiv 1,2,4\pmod 7$ *and so* $p=x^2+7y^2$, *then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 3,5,6\pmod 7$, *then*

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{2k}{k}^3&\equiv\frac{352}{9}(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k}\\&\equiv\begin{cases}-\frac{11}{4}p^2\binom{3[p/7]}{[p/7]}^{-2}\equiv-11p^2\binom{[3p/7]}{[p/7]}^{-2}\pmod{p^3}&\text{if } p\equiv 3\pmod 7,\\-\frac{99}{64}p^2\binom{3[p/7]}{[p/7]}^{-2}\equiv-11p^2\binom{[6p/7]}{[2p/7]}^{-2}\pmod{p^3}&\text{if } p\equiv 5\pmod 7,\\-\frac{25}{176}p^2\binom{3[p/7]}{[p/7]}^{-2}\equiv-11p^2\binom{[3p/7]}{[p/7]+1}^{-2}\pmod{p^3}&\text{if } p\equiv 6\pmod 7.\end{cases}\end{aligned}$$

Remark 4.4 From [5, Theorems 9.2.6, 12.9.8 and 12.9.9] we know that if p is an odd prime such that $p\equiv 1,2,4\pmod 7$ and so $p=x^2+7y^2$, in 1848 Eisenstein proved that

$$\binom{3[\frac p7]}{[\frac p7]}\equiv\begin{cases}2x\pmod p&\text{if } p\equiv 1\pmod 7\text{ and }x\equiv 1\pmod 7,\\2x\pmod p&\text{if } p\equiv 2\pmod 7\text{ and }x\equiv 3\pmod 7,\\\frac25x\pmod p&\text{if } p\equiv 4\pmod 7\text{ and }x\equiv 2\pmod 7.\end{cases}$$

Conjecture 4.9. *Let* $p>3$ *be a prime.*
(i) *If* $p\equiv 1\pmod 3$ *and so* $p=x^2+3y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 2\pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv-8(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}\equiv-p^2\binom{\frac{p-1}{2}}{\frac{p-5}{6}}^{-2}\pmod{p^3}.$$

Conjecture 4.10. *Let* p *be an odd prime.*
(i) *If* $p\equiv 1\pmod 4$ *and so* $p=x^2+y^2$ *with* $2\nmid x$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv(-1)^{\frac{p-1}{4}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 3\pmod 4$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv-3\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv 6(-1)^{\frac{p+1}{4}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k}\equiv\frac34p^2\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2}\pmod{p^3}.$$

Conjecture 4.11. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\begin{cases}(-1)^{\frac{p-1}{2}}(4x^2-2p-\frac{p^2}{4x^2}) \pmod{p^3} & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+2y^2,\\ \frac{p^2}{3}\binom{[p/4]}{[p/8]}^{-2} \pmod{p^3} & \text{if } p\equiv 5 \pmod 8,\\ \frac{3}{2}p^2\binom{[p/4]}{[p/8]}^{-2} \pmod{p^3} & \text{if } p\equiv 7 \pmod 8.\end{cases}$$

Conjecture 4.12. *Let p be a prime with $p>3$. Then*

$$\sum_{n=0}^{p-1}A_n\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+2y^2,\\ \frac{17}{27}p^2\binom{[p/4]}{[p/8]}^{-2} \pmod{p^3} & \text{if } p\equiv 5 \pmod 8,\\ -\frac{17}{6}p^2\binom{[p/4]}{[p/8]}^{-2} \pmod{p^3} & \text{if } p\equiv 7 \pmod 8\end{cases}$$

and

$$\sum_{n=0}^{p-1}(-1)^nA_n\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ \frac{5}{4}p^2\binom{\frac{p-1}{2}}{\frac{p-5}{6}}^{-2} \pmod{p^3} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Remark 4.5 Z.W. Sun [36] conjectured the congruences for $\sum_{n=0}^{p-1}A_n$ and $\sum_{n=0}^{p-1}(-1)^nA_n$ modulo p^2 and proved the congruences when the modulus is p. He also conjectured that

$$\begin{aligned}\sum_{k=0}^{p-1}A_k&\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \pmod{p^3}\quad\text{for}\quad p\equiv 1,3 \pmod 8,\\ \sum_{k=0}^{p-1}(-1)^kA_k&\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}\quad\text{for}\quad p\equiv 1 \pmod 3.\end{aligned}$$

Let $p>3$ be a prime. In [39] Z.W. Sun conjectured the congruence for $\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}^4 \pmod{p^2}$, and posed conjectures on $\sum_{n=0}^{p-1}\frac{D_n}{m^n} \pmod{p^2}$ for $m=1,-2,4,-8,8,16,-32,64$. In [30] the author proved some congruences for $\sum_{n=0}^{p-1}\frac{D_n}{m^n}$ and $\sum_{n=0}^{p-1}\frac{b_n}{m^n}$ modulo p. Now we present congruences for such sums modulo p^3.

Conjecture 4.13. *Let p be a prime with $p\neq 2,3,13,47$.*
(i) *If $p\equiv 1,3 \pmod 8$ and so $p=x^2+2y^2$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}\equiv\sum_{n=0}^{p-1}b_n\equiv\sum_{n=0}^{p-1}\frac{b_n}{81^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{8^n}\equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.$$

(ii) *If* $p\equiv 5,7\pmod 8$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}\equiv -\frac{33}{47}\sum_{n=0}^{p-1}b_n\equiv -\frac{99}{13}\sum_{n=0}^{p-1}\frac{b_n}{81^n}\equiv 33\sum_{n=0}^{p-1}\frac{D_n}{8^n}$$
$$\equiv\begin{cases}\frac{11}{9}p^2\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 8,\\ -\frac{11}{2}p^2\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod 8.\end{cases}$$

Conjecture 4.14. *Let* p *be a prime with* $p\neq 2,3,5,13$. *If* $p\equiv 1,17,19,23\pmod{30}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}}\equiv\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}^4\equiv\sum_{n=0}^{p-1}D_n\equiv\sum_{n=0}^{p-1}\frac{D_n}{64^n}$$
$$\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } p\equiv 1,19\pmod{30}\text{ and so } p=x^2+15y^2,\\ 2p-12x^2+\frac{p^2}{12x^2}\pmod{p^3} & \text{if } p\equiv 17,23\pmod{30}\text{ and so } p=3x^2+5y^2.\end{cases}$$

If $p\equiv 7,11,13,29\pmod{30}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\equiv 28\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}}\equiv\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}^4\equiv\frac{28}{53}\sum_{n=0}^{p-1}D_n\equiv -\frac{112}{13}\sum_{n=0}^{p-1}\frac{D_n}{64^n}$$
$$\equiv\begin{cases}\frac{7}{2}p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod{30},\\ 14p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \text{if } p\equiv 11\pmod{30},\\ \frac{7}{32}p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \text{if } p\equiv 13\pmod{30},\\ \frac{7}{8}p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \text{if } p\equiv 29\pmod{30}.\end{cases}$$

Conjecture 4.15. *Let* $p>3$ *be a prime.*
(i) *If* $p\equiv 1\pmod 3$ *and so* $p=x^2+3y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\sum_{n=0}^{p-1}\binom{2n}{n}\frac{f_n}{(-4)^n}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k}\equiv\sum_{n=0}^{p-1}\frac{D_n}{(-2)^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{16^n}$$
$$\equiv\sum_{n=0}^{p-1}\frac{D_n}{(-32)^n}\equiv\sum_{n=0}^{p-1}\frac{b_n}{(-9)^n}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 2\pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv -\sum_{n=0}^{p-1}\binom{2n}{n}\frac{f_n}{(-4)^n}\equiv -\frac{1}{4}\sum_{n=0}^{p-1}\frac{D_n}{(-2)^n}\equiv -\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv 2\sum_{n=0}^{p-1}\frac{D_n}{16^n}$$

$$\equiv -\sum_{n=0}^{p-1}\frac{D_n}{(-32)^n} \equiv -2\sum_{n=0}^{p-1}\frac{b_n}{(-9)^n} \equiv -\frac{p^2}{2}\binom{\frac{p-1}{2}}{\frac{p-5}{6}}^{-2} \pmod{p^3}.$$

Conjecture 4.16. *Let $p>3$ be a prime.*
(i) *If $p\equiv 1 \pmod 4$, then*

$$\sum_{n=0}^{p-1}\frac{b_n}{(-3)^n} \equiv \sum_{n=0}^{p-1}\frac{b_n}{(-27)^n} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k}$$
$$\equiv \begin{cases} 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p\equiv 1 \pmod{12} \text{ and so } p=x^2+9y^2,\\ 2p-2x^2+\frac{p^2}{2x^2} \pmod{p^3} & \text{if } p\equiv 5 \pmod{12} \text{ and so } 2p=x^2+9y^2.\end{cases}$$

(ii) *If $p\equiv 3 \pmod 4$, then*

$$\sum_{n=0}^{p-1}\frac{b_n}{(-3)^n} \equiv -15\sum_{n=0}^{p-1}\frac{b_n}{(-27)^n} \equiv 10\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k}$$
$$\equiv \begin{cases} -\frac{5}{3}p^2\binom{[p/3]}{[p/12]}^{-2} \pmod{p^3} & \text{if } p\equiv 7 \pmod{12},\\ \frac{5}{6}p^2\binom{[p/3]}{[p/12]}^{-2} \pmod{p^3} & \text{if } p\equiv 11 \pmod{12}.\end{cases}$$

Conjecture 4.17. *Let p be a prime with $p>7$.*
(i) *If $p\equiv 1,2,4 \pmod 7$ and so $p=x^2+7y^2$, then*

$$\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{W_n}{(-27)^n} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-3969)^k}$$
$$\equiv \left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.$$

(ii) *If $p\equiv 3,5,6 \pmod 7$, then*

$$\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{W_n}{(-27)^n} \equiv -\frac{9}{40}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k} \equiv \frac{45}{28}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-3969)^k}$$
$$\equiv -\frac{375}{752}\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}} \equiv \begin{cases} \frac{5}{16}p^2\binom{3[p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } p\equiv 3 \pmod 7,\\ \frac{45}{256}p^2\binom{3[p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } p\equiv 5 \pmod 7,\\ \frac{125}{7744}p^2\binom{3[p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } p\equiv 6 \pmod 7.\end{cases}$$

Conjecture 4.18. *Let p be a prime with $p>7$ and $p\neq 71$.*
(i) *If $p\equiv 1,3 \pmod 8$ and so $p=x^2+2y^2$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \equiv \left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}} \equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.$$

(ii) *If* $p\equiv 5,7\pmod 8$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv-\frac{441}{71}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}\equiv-\frac{25}{7}\left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}}$$
$$\equiv\begin{cases}\frac{p^2}{3}\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 8,\\ -\frac{3}{2}p^2\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod 8.\end{cases}$$

Conjecture 4.19. *Let p be a prime with $p>5$.*
(i) *If* $p\equiv 1\pmod 3$ *and so* $p=x^2+3y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k}\equiv\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^k}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 2\pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k}\equiv 25\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^k}\equiv p^2\binom{(p-1)/2}{(p-5)/6}^{-2}\pmod{p^3}.$$

Conjecture 4.20. *Let p be a prime with $p\neq 2,3,11$.*
(i) *If* $p\equiv 1\pmod 4$ *and so* $p=x^2+y^2$ *with* $2\nmid x$,

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\left(\frac{33}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{648^k}$$
$$\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{54^k}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 3\pmod 4$, *then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\frac{121}{13}\left(\frac{33}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}}\equiv-3\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{648^k}$$
$$\equiv-\frac{5}{3}\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{54^k}\equiv\frac{5}{12}p^2\binom{(p-3)/2}{(p-3)/4}^{-2}\pmod{p^3}.$$

Conjecture 4.21. *Let p be a prime with $p>5$.*
(i) *If* $p\equiv 1\pmod 3$ *and so* $4p=L^2+27M^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\left(\frac{10}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k}\equiv L^2-2p-\frac{p^2}{L^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 2\pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\frac{800}{161}\left(\frac{10}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k}$$

$$\equiv \begin{cases} 3p^2\binom{[2p/3]}{[p/12]}^{-2} \equiv \frac{3}{4}p^2\binom{[2p/3]}{[p/3]}^{-2} \pmod{p^3} & \text{if } 12 \mid p-5, \\ \frac{3}{49}p^2\binom{[2p/3]}{[p/12]}^{-2} \equiv \frac{3}{4}p^2\binom{[2p/3]}{[p/3]}^{-2} \pmod{p^3} & \text{if } 12 \mid p-11. \end{cases}$$

Conjecture 4.22. *Let p be a prime with $p \neq 2, 11, 13$.*
(i) *If $p \equiv 1, 3, 4, 5, 9 \pmod{11}$ and so $4p = u^2 + 11v^2$ with $u, v \in \mathbb{Z}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k} \equiv \left(\frac{-2}{p}\right)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}} \equiv u^2 - 2p - \frac{p^2}{u^2} \pmod{p^3}.$$

(ii) *If $p \equiv 2, 6, 7, 8, 10 \pmod{11}$ and $f = [\frac{p}{11}]$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k} \equiv \frac{160}{39}\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}}$$

$$\equiv \begin{cases} -p^2\Big(\frac{5\binom{4f}{2f}}{2\binom{3f}{f}\binom{6f}{3f}}\Big)^2 \pmod{p^3} & \text{if } p \equiv 2 \pmod{11}, \\ -p^2\Big(\frac{13\binom{4f}{2f}}{30\binom{3f}{f}\binom{6f}{3f}}\Big)^2 \pmod{p^3} & \text{if } p \equiv 6 \pmod{11}, \\ -p^2\Big(\frac{85\binom{4f}{2f}}{558\binom{3f}{f}\binom{6f}{3f}}\Big)^2 \pmod{p^3} & \text{if } p \equiv 7 \pmod{11}, \\ -p^2\Big(\frac{7\binom{4f}{2f}}{148\binom{3f}{f}\binom{6f}{3f}}\Big)^2 \pmod{p^3} & \text{if } p \equiv 8 \pmod{11}, \\ -p^2\Big(\frac{29\binom{4f}{2f}}{756\binom{3f}{f}\binom{6f}{3f}}\Big)^2 \pmod{p^3} & \text{if } p \equiv 10 \pmod{11}. \end{cases}$$

Conjecture 4.23. *Let p be a prime with $p \neq 2, 3, 19$.*
(i) *If $p \equiv 1, 4, 5, 6, 7, 9, 11, 16, 17 \pmod{19}$ and so $4p = u^2 + 19v^2$ with $u, v \in \mathbb{Z}$, then*

$$\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}} \equiv u^2 - 2p - \frac{p^2}{u^2} \pmod{p^3}.$$

(ii) *If $p \equiv 2, 3 \pmod{19}$ and $f = [\frac{p}{19}]$, then*

$$\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}} \equiv \begin{cases} -\frac{985}{384}p^2\Big(\frac{\binom{6f}{3f}\binom{10f}{2f}}{\binom{6f}{f}\binom{10f}{3f}\binom{10f}{4f}}\Big)^2 \pmod{p^3} & \text{if } 19 \mid p-2, \\ -\frac{197}{58080}p^2\Big(\frac{\binom{6f}{3f}\binom{10f}{2f}}{\binom{6f}{f}\binom{10f}{3f}\binom{10f}{4f}}\Big)^2 \pmod{p^3} & \text{if } 19 \mid p-3. \end{cases}$$

Conjecture 4.24. *Let p be a prime with $p \neq 2, 3, 23$.*
(i) *If $p \equiv 1, 5, 7, 11 \pmod{24}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k} \equiv \left(\frac{p}{3}\right)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}} \equiv \left(\frac{p}{3}\right)\sum_{n=0}^{p-1} \frac{b_n}{9^n} \equiv \sum_{n=0}^{p-1} \frac{D_n}{(-8)^n}$$

$$\equiv \begin{cases} 4x^2-2p-\dfrac{p^2}{4x^2} \pmod{p^3} & \text{if } p\equiv 1,7 \pmod{24} \text{ and so } p=x^2+6y^2,\\ 8x^2-2p-\dfrac{p^2}{8x^2} \pmod{p^3} & \text{if } p\equiv 5,11 \pmod{24} \text{ and so } p=2x^2+3y^2. \end{cases}$$

(ii) *If* $p\equiv 13,17,19,23 \pmod{24}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}\equiv -7\Big(\frac{p}{3}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\equiv \frac{7}{23}\Big(\frac{p}{3}\Big)\sum_{n=0}^{p-1}\frac{b_n}{9^n}\equiv -\frac{7}{25}\sum_{n=0}^{p-1}\frac{D_n}{(-8)^n} \pmod{p^3}.$$

Remark 4.6 Suppose that p is a prime such that $p\equiv 1,5,7,11 \pmod{24}$. In [35] Z.W. Sun conjectured that $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/216^k\equiv\big(\frac{p}{3}\big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}/48^{2k} \pmod{p^3}$.

Conjecture 4.25. *Let p be a prime such that $p>3$ and $p\equiv 3,5,6 \pmod 7$. Then*

$$\Big(\frac{p}{3}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{63^k}\equiv\begin{cases}\dfrac{p}{3\binom{3[p/7]}{[p/7]}} \pmod{p^2} & \text{if } p\equiv 3 \pmod 7,\\ -\dfrac{p}{4\binom{3[p/7]}{[p/7]}} \pmod{p^2} & \text{if } p\equiv 5 \pmod 7,\\ \dfrac{5p}{66\binom{3[p/7]}{[p/7]}} \pmod{p^2} & \text{if } p\equiv 6 \pmod 7.\end{cases}$$

Remark 4.7 For the conjecture for $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}/63^k \pmod{p^2}$ in the cases $p\equiv 1,2,4 \pmod 7$ see [35, Conjecture 5.14]. In [27] the author showed that these congruences are true when the modulus is p. For any prime $p>3$, in [35] Z.W. Sun also made conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}/48^k$ and $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}/72^k$ modulo p^2.

Conjecture 4.26. *Let p be a prime with $p\equiv 1 \pmod 3$. Then*

$$\sum_{k=1}^{p-1}\frac{kW_k}{(-9)^k}\equiv 0 \pmod{p^2}.$$

Conjecture 4.27. *Let p be a prime with $p>3$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(-12)^k}$$
$$\equiv\begin{cases} L^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } 4p=L^2+27M^2 \text{ with } L,M\in\mathbb{Z},\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 4.28. *Let p be an odd prime, $n\in\{-640320,-5280,-960,-96,-32,20,255\}$ and $n(n-12)\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{W_k}{(n-12)^k}\equiv\Big(\frac{n(n-12)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}} \pmod{p^2}.$$

Hence the congruences in Theorems 2.4-2.10 also hold modulo p^2.

Conjecture 4.29. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{3k+1}{(-16)^k}f_k \equiv (-1)^{\frac{p-1}{2}}p + p^3E_{p-3} \pmod{p^4},$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{7k+2}{(-27)^k}W_k \equiv 2\Big(\frac{-3}{p}\Big)p - 4p^3U_{p-3} \pmod{p^4},$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{7k+3}{8^k}W_k \equiv 3\Big(\frac{-2}{p}\Big)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{7k+2}{54^k}W_k \equiv 2\Big(\frac{-3}{p}\Big)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{14k+3}{(-44)^k}W_k \equiv 3\Big(\frac{-11}{p}\Big)p \pmod{p^2} \quad \textit{for} \quad p \neq 11,$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{38k+7}{(-108)^k}W_k \equiv 7\Big(\frac{-3}{p}\Big)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{133k+26}{243^k}W_k \equiv 26\Big(\frac{-3}{p}\Big)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{602k+85}{(-972)^k}W_k \equiv 85\Big(\frac{-3}{p}\Big)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{4154k+481}{(-5292)^k}W_k \equiv 481\Big(\frac{-3}{p}\Big)p \pmod{p^2} \quad \textit{for } p \neq 7.$$

Remark 4.8 In [12] Guo proved that for any odd prime p,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{3k+1}{(-16)^k}f_k \equiv (-1)^{\frac{p-1}{2}}p \pmod{p^3}.$$

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