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Some relations between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$

by

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. Let $\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{N}^4 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. For $n \in \mathbb{N}$ let

$$\sigma(n) = \sum_{d\mid n, d\in\mathbb{N}} d.$$

For $a,b,c,d \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$ set

$$N(a,b,c,d;n) = \big|\{(x,y,z,w) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dw^2\}\big|$$

and

$$t(a,b,c,d;n) = \Big|\Big\{(x,y,z,w) \in \mathbb{Z}^4 \mid n = a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2} + c\frac{z(z-1)}{2} + d\frac{w(w-1)}{2}\Big\}\Big|.$$

The numbers $\frac{x(x-1)}{2}$ ($x \in \mathbb{Z}$) are called triangular numbers.

In 1828 Jacobi showed that

$$N(1,1,1,1;n) = 8\sum_{d\mid n, 4\nmid d} d.$$

For $d \in \{3,5\}$, in 1847 Eisenstein (see [13]) gave formulas for the number of proper representations of n by $x^2 + y^2 + z^2 + dw^2$ (assuming that $gcd(x,y,z,w) = 1$). From 1859 to 1866 Liouville made about 90 conjectures on $N(a,b,c,d;n)$ in a series of papers. Most conjectures of Liouville have been proved. See [2-9], Cooper's survey paper [12], Dickson's historical comments [13] and Williams' book [18].

Let

$$t'(a,b,c,d;n) = \Big|\Big\{(x,y,z,w) \in \mathbb{N}^4 \mid n = a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2}$$

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$$+c\frac{z(z-1)}{2}+d\frac{w(w-1)}{2}\Big\}\Big|.$$

As $\frac{x(x-1)}{2}=\frac{(-x+1)(-x)}{2}$ we have

$$t(a,b,c,d;n)=16t'(a,b,c,d;n). \tag{1.1}$$

In [14] Legendre stated that

$$t'(1,1,1,1;n)=\sigma(2n+1). \tag{1.2}$$

In 2003, Williams [16] showed that

$$t'(1,1,2,2;n)=\frac{1}{4}\sum_{d\mid 4n+3}\big(d-(-1)^{\frac{d-1}{2}}\big). \tag{1.3}$$

For $a,b,c,d\in\mathbb{N}$ with $5\le a+b+c+d\le 8$ let

$$C(a,b,c,d)=16+4i_1(i_1-1)i_2+8i_1i_3,$$

where i_j is the number of elements in $\{a,b,c,d\}$ which are equal to j. When $a+b+c+d\in\{5,6,7\}$, in 2005 Adiga, Cooper and Han [1] showed that

$$C(a,b,c,d)t'(a,b,c,d;n)=N(a,b,c,d;8n+a+b+c+d). \tag{1.4}$$

When $a+b+c+d=8$, in 2008 Baruah, Cooper and Hirschhorn [10] proved that

$$\begin{aligned}&C(a,b,c,d)t'(a,b,c,d;n)\\&=N(a,b,c,d;8n+8)-N(a,b,c,d;2n+2).\end{aligned} \tag{1.5}$$

In 2009, Cooper [12] determined $t'(a,b,c,d;n)$ for $(a,b,c,d)=(1,1,1,3),\ (1,3,3,3),\ (1,2,2,3),\ (1,3,6,6),\ (1,3,4,4),\ (1,1,2,6)$ and $(1,3,\ 12,12)$.

In [15], Wang and Sun obtained explicit formulas for $t(a,b,c,d;n)$ in the cases $(a,b,c,d)=(1,2,2,4),\ (1,2,4,4),\ (1,1,4,4),\ (1,4,4,4),\ (1,3,3,9),\ (1,1,9,9),\ (1,9,9,\ 9),\ (1,1,1,9),\ (1,3,9,9)$ and $(1,1,3,9)$.

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}=1+2\sum_{n=1}^{\infty}q^{n^2}\quad\text{and}\quad\psi(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}\ (|q|<1).$$

It is evident that for $|q|<1$,

$$\sum_{n=0}^{\infty}N(a,b,c,d;n)q^n=\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d), \tag{1.6}$$

$$\sum_{n=0}^{\infty}t'(a,b,c,d;n)q^n=\psi(q^a)\psi(q^b)\psi(q^c)\psi(q^d). \tag{1.7}$$

From [10, Lemma 4.1] or [11] we know that for $|q| < 1$,

$$\psi(q)^2 = \varphi(q)\psi(q^2), \tag{1.8}$$
$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{1.9}$$
$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2, \tag{1.10}$$
$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}). \tag{1.11}$$

By (1.9), for $k \in \mathbb{N}$,

$$\begin{aligned}\varphi(q^k) &= \varphi(q^{4k}) + 2q^k\psi(q^{8k}) \\ &= \varphi(q^{16k}) + 2q^{4k}\psi(q^{32k}) + 2q^k\psi(q^{8k}).\end{aligned} \tag{1.12}$$

In this paper, using (1.6)-(1.12) we reveal some connections between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$. Suppose $k,m \in \{0,1,2,\ldots\}$, $a,n \in \mathbb{N}$ and $2 \nmid a$. We show that

$$\begin{aligned}t(a,b,c,d;n) = \frac{2}{3}\big(&N(a,b,c,d;8n+a+b+c+d) \\ &- N(a,b,c,d;2n+(a+b+c+d)/4)\big)\end{aligned}$$

for $(a,b,c,d) = (a,a,2a,8m+4)$ and $(a,3a,4k+2,4m+2)$ with $k \equiv m \pmod 2$. For $2 \nmid ak$ we show that

$$t(a,3a,k,k;m) = \frac{2}{3}N(a,3a,2k,2k;8m+4a+2k).$$

For $n \equiv k + \frac{a-1}{2} \pmod 2$, we prove that

$$t(a,3a,8k+4,4m+2;n) = \frac{2}{3}N(a,3a,8k+4,4m+2;8n+4m+8k+4a+6).$$

Let $a,k,m \in \mathbb{N}$ with $2 \nmid a$. We also show that

$$\begin{aligned}&t(a,a,2a,4k;4m+3a) = 4t(a,2a,4a,k;m), \\ &t(a,a,6a,4k;4m+3a) = 2t(a,a,6a,k;m), \\ &t(a,a,8a,2k;2m) = t(a,2a,2a,k;m), \\ &t(a,a,8a,2k;2m+a) = 2t(a,4a,4a,k;m).\end{aligned}$$

In addition, we give explicit formulas for $t(1,3,3,6;n)$, $t(1,1,8,8;n)$ and $t(1,1,4,8;n)$. We also pose many conjectures on the relations between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$.

2. Main results

Lemma 2.1. *Suppose $a,k,m,n \in \mathbb{N}$ and $2 \nmid a$. Then*

$$N(a,a,2k,2m;2n) = N(a,a,k,m;n).$$

Proof. Suppose $|q|<1$. Using (1.10) we see that

$$\sum_{n=0}^{\infty} N(a,a,2k,2m;n)q^n$$
$$= \varphi(q^a)^2\varphi(q^{2k})\varphi(q^{2m}) = \big(\varphi(q^{2a})^2 + 4q^a\psi(q^{4a})^2\big)\varphi(q^{2k})\varphi(q^{2m}).$$

Extracting the even powers we obtain

$$\sum_{n=0}^{\infty} N(a,a,2k,2m;2n)q^{2n} = \varphi(q^{2a})^2\varphi(q^{2k})\varphi(q^{2m}).$$

Replacing q with $q^{1/2}$ in the above formula we obtain

$$\sum_{n=0}^{\infty} N(a,a,2k,2m;2n)q^n$$
$$= \varphi(q^a)^2\varphi(q^k)\varphi(q^m) = \sum_{n=0}^{\infty} N(a,a,k,m;n)q^n.$$

Comparing the coefficients of q^n on both sides we obtain the result.

Remark 2.1 In the case $a=1$, Lemma 2.1 has been given in [5, Proposition 4.1(iii)].

Lemma 2.2. *For $|q|<1$ we have*

$$\varphi(q)^3 = \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^4)^2 + 12q^2\psi(q^4)^2\psi(q^8) + 8q^3\psi(q^8)^3.$$

Proof. By (1.8) and (1.9),

$$\begin{aligned}\varphi(q)^3 &= (\varphi(q^4) + 2q\psi(q^8))^3\\ &= \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^8)(\varphi(q^4) + 2q\psi(q^8)) + 8q^3\psi(q^8)^3\\ &= \varphi(q^4)^3 + 6q\psi(q^4)^2(\varphi(q^4) + 2q\psi(q^8)) + 8q^3\psi(q^8)^3.\end{aligned}$$

This yields the result.

Theorem 2.1. *Let $a\in\{1,3,5,\ldots\}$ and $m\in\{0,1,2,\ldots\}$. For $n\in\mathbb{N}$ we have*

$$\begin{aligned}t(a,a,2a,8m+4;n) &= \frac{2}{3}(N(a,a,a,4m+2;4n+4m+2a+2)\\ &\quad - N(a,a,a,4m+2;n+m+(a+1)/2))\\ &= \frac{2}{3}(N(a,a,2a,8m+4;8n+8m+4a+4)\\ &\quad - N(a,a,2a,8m+4;2n+2m+a+1)).\end{aligned}$$

Proof. Suppose $|q|<1$. By (1.6), (1.9) and Lemma 2.2,

$$\begin{aligned}&\sum_{n=0}^{\infty} N(a,a,a,4m+2;n)q^n = \varphi(q^a)^3\varphi(q^{4m+2})\\ &= \big(\varphi(q^{4a})^3 + 6q^a\varphi(q^{4a})\psi(q^{4a})^2 + 12q^{2a}\psi(q^{4a})^2\psi(q^{8a}) + 8q^{3a}\psi(q^{8a})^3\big)\\ &\quad\times\big(\varphi(q^{4(4m+2)}) + 2q^{4m+2}\psi(q^{8(4m+2)})\big)\end{aligned}\tag{2.1}$$

For any $r,s\in\mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^\infty b_nq^{8n}$. Hence from (2.1) we deduce that

$$\begin{aligned}&\sum_{n=0}^\infty N(a,a,a,4m+2;4n)q^{4n}\\&=\varphi(q^{4a})^3\varphi(q^{16m+8})+12q^{2a}\psi(q^{4a})^2\psi(q^{8a})\cdot 2q^{4m+2}\psi(q^{32m+16}).\end{aligned}$$

Replacing q with $q^{1/4}$ in the above formula we obtain

$$\begin{aligned}&\sum_{n=0}^\infty N(a,a,a,4m+2;4n)q^{n}\\&=\varphi(q^{a})^3\varphi(q^{4m+2})+24q^{m+(a+1)/2}\psi(q^{a})^2\psi(q^{2a})\psi(q^{8m+4})\\&=\sum_{n=0}^\infty N(a,a,a,4m+2;n)q^n\\&\quad+24q^{m+(a+1)/2}\sum_{n=0}^\infty t'(a,a,2a,8m+4;n)q^n.\end{aligned}$$

Now comparing the coefficients of $q^{n+m+(a+1)/2}$ on both sides and then applying (1.1) and Lemma 2.1 we obtain

$$\begin{aligned}&\frac{3}{2}t(a,a,2a,8m+4;n)=24t'(a,a,2a,8m+4;n)\\&=N(a,a,a,4m+2;4n+4m+2a+2)\\&\quad-N(a,a,a,4m+2;n+m+(a+1)/2)\\&=N(a,a,2a,8m+4;8n+8m+4a+4)\\&\quad-N(a,a,2a,8m+4;2n+2m+a+1).\end{aligned}$$

This is the result.

Lemma 2.3. *For $|q|<1$ we have*

$$\begin{aligned}\varphi(q)\varphi(q^3)&=\varphi(q^{16})\varphi(q^{48})+4q^{16}\psi(q^{32})\psi(q^{96})+2q\varphi(q^{48})\psi(q^8)\\&\quad+2q^3\varphi(q^{16})\psi(q^{24})+6q^4\psi(q^8)\psi(q^{24})\\&\quad+4q^{13}\psi(q^8)\psi(q^{96})+4q^7\psi(q^{24})\psi(q^{32}).\end{aligned}$$

Proof. By (1.12),

$$\begin{aligned}\varphi(q)\varphi(q^3)&=\big(\varphi(q^{16})+2q^4\psi(q^{32})+2q\psi(q^8)\big)\\&\quad\times\big(\varphi(q^{48})+2q^{12}\psi(q^{96})+2q^3\psi(q^{24})\big)\\&=\varphi(q^{16})\varphi(q^{48})+4q^{16}\psi(q^{32})\psi(q^{96})+2q\varphi(q^{48})\psi(q^8)\\&\quad+2q^3\varphi(q^{16})\psi(q^{24})+2q^4\big(\varphi(q^{48})\psi(q^{32})+q^8\varphi(q^{16})\psi(q^{96})\\&\quad+2\psi(q^8)\psi(q^{24})\big)+4q^{13}\psi(q^8)\psi(q^{96})+4q^7\psi(q^{24})\psi(q^{32}).\end{aligned}$$

Note that $\varphi(q^{48})\psi(q^{32})+q^8\varphi(q^{16})\psi(q^{96})=\psi(q^8)\psi(q^{24})$ by (1.11). Hence we obtain the result.

Theorem 2.2. *Let* $a \in \{1,3,5,\dots\}$, $k,m \in \{0,1,2,\dots\}$ *and* $k \equiv m \pmod 2$. *For* $n \in \mathbb{N}$ *we have*

$$\begin{aligned}&t(a,3a,4k+2,4m+2;n)\\&=\frac{2}{3}(N(a,3a,4k+2,4m+2;8n+4m+4k+4a+4)\\&\quad-N(a,3a,4k+2,4m+2;2n+m+k+a+1)).\end{aligned}$$

Proof. Suppose $|q|<1$. Using Lemma 2.3 and (1.12) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty}N(a,3a,4k+2,4m+2;n)q^n=\varphi(q^a)\varphi(q^{3a})\varphi(q^{4k+2})\varphi(q^{4m+2})\\&=\varphi(q^a)\varphi(q^{3a})\big(\varphi(q^{4(4k+2)})+2q^{4k+2}\psi(q^{8(4k+2)})\big)\\&\quad\times\big(\varphi(q^{4(4m+2)})+2q^{4m+2}\psi(q^{8(4m+2)})\big)\\&=\big(\varphi(q^{16a})\varphi(q^{48a})+4q^{16a}\psi(q^{32a})\psi(q^{96a})+2q^a\varphi(q^{48a})\psi(q^{8a})\\&\quad+2q^{3a}\varphi(q^{16a})\psi(q^{24a})+6q^{4a}\psi(q^{8a})\psi(q^{24a})\\&\quad+4q^{13a}\psi(q^{8a})\psi(q^{96a})+4q^{7a}\psi(q^{24a})\psi(q^{32a})\big)\\&\quad\times\big(\varphi(q^{16k+8})\varphi(q^{16m+8})+2q^{4m+2}\varphi(q^{16k+8})\psi(q^{32m+16})\\&\quad+2q^{4k+2}\psi(q^{32k+16})\varphi(q^{16m+8})+4q^{4k+4m+4}\psi(q^{32k+16})\psi(q^{32m+16})\big).\end{aligned}$$

For any $r,s\in\mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty}b_nq^{8n}$. Thus, from the above we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty}N(a,3a,4k+2,4m+2;8n)q^{8n}\\&=\big(\varphi(q^{16a})\varphi(q^{48a})+4q^{16a}\psi(q^{32a})\psi(q^{96a})\big)\varphi(q^{16k+8})\varphi(q^{16m+8})\\&\quad+24q^{4(k+m+a+1)}\psi(q^{8a})\psi(q^{24a})\psi(q^{32k+16})\psi(q^{32m+16}).\end{aligned}$$

Replacing q with $q^{1/8}$ in the above formula we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}N(a,3a,4k+2,4m+2;8n)q^{n}\\&=(\varphi(q^{2a})\varphi(q^{6a})+4q^{2a}\psi(q^{4a})\psi(q^{12a}))\varphi(q^{2k+1})\varphi(q^{2m+1})\\&\quad+24q^{(k+m+a+1)/2}\psi(q^{a})\psi(q^{3a})\psi(q^{4k+2})\psi(q^{4m+2}).\end{aligned}$$

On the other hand, using (1.9) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty}N(a,3a,4k+2,4m+2;n)q^{n}\\&=\varphi(q^a)\varphi(q^{3a})\varphi(q^{4k+2})\varphi(q^{4m+2})\\&=\big(\varphi(q^{4a})+2q^a\psi(q^{8a})\big)\big(\varphi(q^{12a})+2q^{3a}\psi(q^{24a})\big)\varphi(q^{4k+2})\varphi(q^{4m+2}).\end{aligned}$$

Extracting the even powers we obtain

$$\sum_{n=0}^{\infty} N(a,3a,4k+2,4m+2;2n)q^{2n}$$
$$= \big(\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a})\big)\varphi(q^{4k+2})\varphi(q^{4m+2}).$$

Replacing q with $q^{1/2}$ we then obtain

$$\sum_{n=0}^{\infty} N(a,3a,4k+2,4m+2;2n)q^{n}$$
$$= \big(\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a})\big)\varphi(q^{2k+1})\varphi(q^{2m+1}).$$

Hence

$$\sum_{n=0}^{\infty} (N(a,3a,4k+2,4m+2;8n) - N(a,3a,4k+2,4m+2;2n))q^n$$
$$= 24q^{(k+m+a+1)/2}\psi(q^a)\psi(q^{3a})\psi(q^{4k+2})\psi(q^{4m+2})$$
$$= 24q^{(k+m+a+1)/2}\sum_{n=0}^{\infty} t'(a,3a,4k+2,4m+2;n)q^n$$
$$= \frac{3}{2}q^{(k+m+a+1)/2}\sum_{n=0}^{\infty} t(a,3a,4k+2,4m+2;n)q^n.$$

Comparing the coefficients of $q^{n+(k+m+a+1)/2}$ yields the result.

Theorem 2.3. *Let $a,k \in \mathbb{N}$ with $2 \nmid ak$. For $m \in \mathbb{N}$ we have*

$$t(a,3a,k,k;m) = \frac{2}{3}N(a,3a,2k,2k;8m+4a+2k).$$

Proof. By (1.10),

$$\varphi(q^{2k})^2 = \varphi(q^{4k})^2 + 4q^{2k}\psi(q^{8k})^2$$
$$= \varphi(q^{8k})^2 + 4q^{4k}\psi(q^{16k})^2 + 4q^{2k}\psi(q^{8k})^2.$$

Thus, applying Lemma 2.3 we see that for $|q| < 1$,

$$\sum_{n=0}^{\infty} N(a,3a,2k,2k;n)q^n = \varphi(q^a)\varphi(q^{3a})\varphi(q^{2k})^2$$
$$= \big(\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a})$$
$$+ 2q^a\varphi(q^{48a})\psi(q^{8a}) + 2q^{3a}\varphi(q^{16a})\psi(q^{24a})$$
$$+ 6q^{4a}\psi(q^{8a})\psi(q^{24a}) + 4q^{13a}\psi(q^{8a})\psi(q^{96a}) + 4q^{7a}\psi(q^{24a})\psi(q^{32a})\big)$$
$$\times \big(\varphi(q^{8k})^2 + 4q^{4k}\psi(q^{16k})^2 + 4q^{2k}\psi(q^{8k})^2\big).$$

For any $r,s \in \mathbb{N}$ the power series expansions of $\varphi(q^s)^r$ and $\psi(q^s)^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above we deduce that

$$\sum_{m=0}^{\infty} N(a,3a,2k,2k;8m+4a+2k)q^{8m+4a+2k}$$

$$= 6q^{4a}\psi(q^{8a})\psi(q^{24a}) \cdot 4q^{2k}\psi(q^{8k})^2 = 24q^{4a+2k}\psi(q^{8a})\psi(q^{24a})\psi(q^{8k})^2.$$

Replacing q with $q^{1/8}$ we get

$$\sum_{m=0}^{\infty} N(a,3a,2k,2k;8m+4a+2k)q^m = 24\psi(q^a)\psi(q^{3a})\psi(q^k)^2$$
$$= 24\sum_{m=0}^{\infty} t'(a,3a,k,k;m)q^m = \frac{24}{16}\sum_{m=0}^{\infty} t(a,3a,k,k;m)q^m.$$

Comparing the coefficients of q^m on both sides yields the result.

Corollary 2.1. *For $n \in \mathbb{N}$ we have*

$$N(2,2,3,9;8n+6) = \frac{3}{5}N(1,1,3,9;8n+6).$$

Proof. Taking $a = 3$ and $k = 1$ in Theorem 2.3 we see that $t(1,1,3,9;m) = \frac{2}{3}N(2,2,3,9;8m+14)$. On the other hand, from [15, the proof of Theorem 2.3] we know that $t(1,1,3,9;m) = \frac{2}{5}N(1,1,3,9;\ 8m+14)$. Thus, the result follows.

Theorem 2.4. *Let $a \in \{1,3,5,\dots\}$ and $k,m \in \{0,1,2,\dots\}$. If $n \in \mathbb{N}$ and $n \equiv k + \frac{a-1}{2} \pmod 2$, then*

$$t(a,3a,8k+4,4m+2;n) = \frac{2}{3}N(a,3a,8k+4,4m+2;8n+4m+8k+4a+6).$$

Proof. Suppose $|q| < 1$. Using Lemma 2.3 and (1.12) we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a,3a,8k+4,4m+2;n)q^n \\
&= \varphi(q^a)\varphi(q^{3a})\varphi(q^{8k+4})\varphi(q^{4m+2}) \\
&= \varphi(q^a)\varphi(q^{3a})\big(\varphi(q^{4(8k+4)}) + 2q^{8k+4}\psi(q^{8(8k+4)})\big) \\
&\quad \times \big(\varphi(q^{4(4m+2)}) + 2q^{4m+2}\psi(q^{8(4m+2)})\big) \\
&= \big(\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) \\
&\quad + 2q^a\varphi(q^{48a})\psi(q^{8a}) + 2q^{3a}\varphi(q^{16a})\psi(q^{24a}) \\
&\quad + 6q^{4a}\psi(q^{8a})\psi(q^{24a}) + 4q^{13a}\psi(q^{8a})\psi(q^{96a}) + 4q^{7a}\psi(q^{24a})\psi(q^{32a})\big) \\
&\quad \times \big(\varphi(q^{32k+16})\varphi(q^{16m+8}) + 2q^{4m+2}\varphi(q^{32k+16})\psi(q^{32m+16}) \\
&\quad + 2q^{8k+4}\psi(q^{64k+32})\varphi(q^{16m+8}) + 4q^{8k+4m+6}\psi(q^{64k+32})\psi(q^{32m+16})\big).
\end{aligned}$$

For any $r,s \in \mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above and the fact that $4m+2 \equiv 4-2(-1)^m \pmod 8$ we deduce that

$$\sum_{n=0}^{\infty} N(a,3a,8k+4,4m+2;8n+4-2(-1)^m)q^{8n+4-2(-1)^m}$$

$$= \big(\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a})\big)$$
$$\times 2q^{4m+2}\varphi(q^{32k+16})\psi(q^{32m+16})$$
$$+ 6q^{4a}\psi(q^{8a})\psi(q^{24a}) \cdot 4q^{8k+4m+6}\psi(q^{64k+32})\psi(q^{32m+16})$$

and so

$$\sum_{n=0}^{\infty} N(a,3a,8k+4,4m+2;8n+4-2(-1)^m)q^{8n}$$
$$= 2q^{8[\frac{m}{2}]}\varphi(q^{32k+16})\psi(q^{32m+16})\big(\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a})\big)$$
$$+ 24q^{8(k+[\frac{m}{2}]+\frac{a+1}{2})}\psi(q^{8a})\psi(q^{24a})\psi(q^{64k+32})\psi(q^{32m+16}),$$

where $[x]$ is the greatest integer not exceeding x. Replacing q with $q^{1/8}$ in the above formula we obtain

$$\sum_{n=0}^{\infty} N(a,3a,8k+4,4m+2;8n+4-2(-1)^m)q^{n}$$
$$= 2q^{[\frac{m}{2}]}\varphi(q^{4k+2})\psi(q^{4m+2})\big(\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a})\big)$$
$$+ 24q^{k+[\frac{m}{2}]+\frac{a+1}{2}}\psi(q^{a})\psi(q^{3a})\psi(q^{8k+4})\psi(q^{4m+2}),$$

Suppose that $n \equiv k+\frac{a-1}{2} \pmod 2$. Then $n+k+[\frac{m}{2}]+\frac{a+1}{2} \equiv [\frac{m}{2}]+1 \pmod 2$. Now comparing the coefficients of $q^{n+k+[\frac{m}{2}]+\frac{a+1}{2}}$ in the above expansion we obtain

$$N\Big(a,3a,8k+4,4m+2;8\big(n+k+[\frac{m}{2}]+\frac{a+1}{2}\big)+4-2(-1)^m\Big)$$
$$= 24t'(a,3a,8k+4,4m+2;n) = \frac{3}{2}t(a,3a,8k+4,4m+2;n).$$

This yields the result.

Theorem 2.5. *Let $a,k \in \mathbb{N}$ with $2 \nmid a$. For $m \in \mathbb{N}$ we have*

$$t(a,a,6a,4k;4m+3a) = 2t(a,a,6a,k;m).$$

Proof. Suppose $|q| < 1$. Using (1.8)-(1.12) we see that

$$\sum_{n=0}^{\infty} t'(a,a,6a,4k;n)q^n$$

(2.2)
$$= \psi(q^a)^2\psi(q^{6a})\psi(q^{4k}) = \varphi(q^a)\psi(q^{2a})\psi(q^{6a})\psi(q^{4k})$$
$$= \big(\varphi(q^{4a}) + 2q^a\psi(q^{8a})\big)\big(\varphi(q^{12a})\psi(q^{8a}) + q^{2a}\varphi(q^{4a})\psi(q^{24a})\big)\psi(q^{4k})$$
$$= \big(\varphi(q^{4a})\varphi(q^{12a})\psi(q^{8a}) + 2q^a\psi(q^{8a})^2\varphi(q^{12a}) + q^{2a}\varphi(q^{4a})^2\psi(q^{24a})$$
$$+ 2q^{3a}\varphi(q^{4a})\psi(q^{8a})\psi(q^{24a})\big)\psi(q^{4k}).$$

Collecting the terms of the form q^{4m+3a} we get

$$\sum_{m=0}^{\infty} t'(a,a,6a,4k;4m+3a)q^{4m+3a}$$

$$= 2q^{3a}\varphi(q^{4a})\psi(q^{8a})\psi(q^{24a})\psi(q^{4k}) = 2q^{3a}\psi(q^{4a})^2\psi(q^{24a})\psi(q^{4k}).$$

Replacing q with $q^{1/4}$ we deduce that

$$\sum_{m=0}^{\infty} t'(a,a,6a,4k;4m+3a)q^m$$
$$= 2\psi(q^a)^2\psi(q^{6a})\psi(q^k) = 2\sum_{m=0}^{\infty} t'(a,a,6a,k;m)q^m.$$

Hence

$$t(a,a,6a,4k;4m+3a)$$
$$= 16t'(a,a,6a,4k;4m+3a) = 32t'(a,a,6a,k;m) = 2t(a,a,6a,k;m)$$

as asserted.

Theorem 2.6. *Let $a,k \in \mathbb{N}$ with $2 \nmid a$. For $m \in \mathbb{N}$ we have*

$$t(a,a,2a,4k;4m+3a) = 4t(a,2a,4a,k;m).$$

Proof. Suppose $|q| < 1$. Using (1.8) and (1.12) we see that

$$\sum_{n=0}^{\infty} t'(a,a,2a,4k;n)q^n$$
$$= \psi(q^a)^2\psi(q^{2a})\psi(q^{4k}) = \varphi(q^a)\psi(q^{2a})^2\psi(q^{4k})$$
$$= \varphi(q^a)\varphi(q^{2a})\psi(q^{4a})\psi(q^{4k})$$
$$= \big(\varphi(q^{4a}) + 2q^a\psi(q^{8a})\big)\big(\varphi(q^{8a}) + 2q^{2a}\psi(q^{16a})\big)\psi(q^{4a})\psi(q^{4k})$$
$$= \big(\varphi(q^{4a})\varphi(q^{8a}) + 2q^a\varphi(q^{8a})\psi(q^{8a}) + 2q^{2a}\varphi(q^{4a})\psi(q^{16a})$$
$$+ 4q^{3a}\psi(q^{8a})\psi(q^{16a})\big)\psi(q^{4a})\psi(q^{4k}).$$

Collecting the terms of the form q^{4m+3a} we get

$$\sum_{m=0}^{\infty} t'(a,a,2a,4k;4m+3a)q^{4m+3a} = 4q^{3a}\psi(q^{8a})\psi(q^{16a})\psi(q^{4a})\psi(q^{4k}).$$

Replacing q with $q^{1/4}$ we see that

$$\sum_{m=0}^{\infty} t'(a,a,2a,4k;4m+3a)q^m$$
$$= 4\psi(q^a)\psi(q^{2a})\psi(q^{4a})\psi(q^k) = 4\sum_{m=0}^{\infty} t'(a,2a,4a,k;m)q^m.$$

Hence

$$t(a,a,2a,4k;4m+3a) = 16t'(a,a,2a,4k;4m+3a)$$

$$= 64t'(a,2a,4a,k;m) = 4t(a,2a,4a,k;m),$$

which completes the proof.

Theorem 2.7. *Let $a,k\in\mathbb{N}$ with $2\nmid a$. For $n\in\mathbb{N}$ we have*

$$t(a,a,8a,2k;2n) = t(a,2a,2a,k;n)$$

and

$$t(a,a,8a,2k;2n+a) = 2t(a,4a,4a,k;n).$$

Proof. Suppose $|q|<1$. Using (1.8) and (1.9) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty} t'(a,a,8a,2k;n)q^n\\&= \psi(q^a)^2\psi(q^{8a})\psi(q^{2k}) = \varphi(q^a)\psi(q^{2a})\psi(q^{8a})\psi(q^{2k})\\&= \psi(q^{2a})(\varphi(q^{4a})+2q^a\psi(q^{8a}))\psi(q^{8a})\psi(q^{2k}).\end{aligned} \tag{2.3}$$

Extracting the even powers and then applying (1.8) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty} t'(a,a,8a,2k;2n)q^{2n}\\&= \psi(q^{2a})\varphi(q^{4a})\psi(q^{8a})\psi(q^{2k}) = \psi(q^{2a})\psi(q^{4a})^2\psi(q^{2k}).\end{aligned}$$

Replacing q with $q^{1/2}$ we then get

$$\sum_{n=0}^{\infty} t'(a,a,8a,2k;2n)q^n = \psi(q^a)\psi(q^{2a})^2\psi(q^k) = \sum_{n=0}^{\infty} t'(a,2a,2a,k;n)q^n.$$

Hence

$$\begin{aligned}&t(a,a,8a,2k;2n)\\&= 16t'(a,a,8a,2k;2n) = 16t'(a,2a,2a,k;n) = t(a,2a,2a,k;n).\end{aligned}$$

On the other hand, extracting the odd powers in (2.3) we see that

$$\sum_{n=0}^{\infty} t'(a,a,8a,2k;2n+a)q^{2n+a} = \psi(q^{2a})\cdot 2q^a\psi(q^{8a})\psi(q^{8a})\psi(q^{2k}).$$

Replacing q with $q^{1/2}$ we then get

$$\begin{aligned}&\sum_{n=0}^{\infty} t'(a,a,8a,2k;2n+a)q^n\\&= 2\psi(q^a)\psi(q^{4a})^2\psi(q^k) = \sum_{n=0}^{\infty} 2t'(a,4a,4a,k;n)q^n.\end{aligned}$$

Hence

$$\begin{aligned}&t(a,a,8a,2k;2n+a)\\&=16t'(a,a,8a,2k;2n+a)=32t'(a,4a,4a,k;n)=2t(a,4a,4a,k;n).\end{aligned}$$

We are done.

Theorem 2.8. *For $n\in\mathbb{N}$ we have*

$$t(1,1,8,8;n)=\sigma(4n+9)-(2-(-1)^n)\sum_{\substack{(x,y)\in\mathbb{Z}\times\mathbb{Z},x\equiv 1 \pmod 4\\ 4n+9=x^2+4y^2}}x.$$

Proof. By Theorem 2.7, $t(1,1,8,8;n)=t(1,2,2,4;n/2)$ for even n, and $t(1,1,8,8;n)=2t(1,4,4,4;(n-1)/2)$ for odd n. Now applying [15, Theorems 3.2 and 3.3] yields the result.

Theorem 2.9. *For $n\in\mathbb{N}$ we have*

$$\begin{aligned}&t(1,1,4,8;n)\\&=2(-1)^n\sum_{d\mid 4n+7}d\Big(\frac{2}{d}\Big)-(1-(-1)^n)\sum_{\substack{(x,y)\in\mathbb{Z}\times\mathbb{Z},x\equiv 1 \pmod 4\\ 4n+7=x^2+2y^2}}x.\end{aligned}$$

Proof. By Theorem 2.7, $t(1,1,4,8;n)=t(1,2,2,2;n/2)$ for even n, and $t(1,1,4,8;n)=2t(1,2,4,4;(n-1)/2)$ for odd n. It is easily seen that

$$\begin{aligned}&t(1,2,2,2;m)\\&=\big|\big\{(x,y,z,w)\in\mathbb{Z}^4\ \big|\ 8m+7=x^2+2y^2+2z^2+2w^2,\ 2\nmid xyzw\big\}\big|\\&=\big|\big\{(x,y,z,w)\in\mathbb{Z}^4\ \big|\ 8m+7=x^2+2y^2+2z^2+2w^2\big\}\big|\\&=N(1,2,2,2;8m+7).\end{aligned}$$

Now applying [17, (1.4)] and [15, Theorem 3.4] yields the result.

Lemma 2.4. *For $|q|<1$ we have*

$$\begin{aligned}\varphi(q)^2=\varphi(q^8)^2+4q^4\psi(q^{16})^2+4q^2\psi(q^8)^2\\+4q\varphi(q^{16})\psi(q^8)+8q^5\psi(q^8)\psi(q^{32}).\end{aligned}$$

Proof. By (1.10),

$$\begin{aligned}&\varphi(q)^2-\varphi(q^8)^2\\&=(\varphi(q)^2-\varphi(q^2)^2)+(\varphi(q^2)^2-\varphi(q^4)^2)+(\varphi(q^4)^2-\varphi(q^8)^2)\\&=4q\psi(q^4)^2+4q^2\psi(q^8)^2+4q^4\psi(q^{16})^2.\end{aligned}$$

By (1.8) and (1.9),

$$\psi(q^4)^2=\varphi(q^4)\psi(q^8)=(\varphi(q^{16})+2q^4\psi(q^{32}))\psi(q^8).$$

Now combining the above we deduce the result.

Theorem 2.10. *If $n \in \mathbb{N}$ and $8n+13 = 3^\beta n_1$ with $n_1 \in \mathbb{N}$ and $3 \nmid n_1$, then*

$$\begin{aligned} t(1,3,3,6;n) &= \frac{2}{5} N(1,3,3,6;8n+13) \\ &= \frac{2}{3}\Big(3^\beta + \Big(\frac{n_1}{3}\Big)\Big) \prod_{p \mid n_1} \frac{p^{\mathrm{ord}_p n_1 + 1} - (\frac{6}{p})^{\mathrm{ord}_p n_1 + 1}}{p - (\frac{6}{p})}, \end{aligned}$$

where p runs all distinct prime divisors of n_1, $(\frac{a}{p})$ is the Legendre symbol and $\mathrm{ord}_p n_1$ is the unique nonnegative integer r such that $p^r \mid n_1$ but $p^{r+1} \nmid n_1$.

Proof. Suppose $|q| < 1$. By (1.12) and Lemma 2.4,

$$\begin{aligned} &\sum_{n=0}^{\infty} N(1,3,3,6;n) q^n = \varphi(q)\varphi(q^6)\varphi(q^3)^2 \\ &\quad = (\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8))(\varphi(q^{24}) + 2q^6\psi(q^{48})) \\ &\quad\quad \times \big(\varphi(q^{24})^2 + 4q^{12}\psi(q^{48})^2 + 4q^6\psi(q^{24})^2 \\ &\quad\quad + 4q^3\varphi(q^{48})\psi(q^{24}) + 8q^{15}\psi(q^{24})\psi(q^{96})\big) \\ &\quad = \big(\varphi(q^{16})\varphi(q^{24}) + 2q^4\psi(q^{32})\varphi(q^{24}) + 2q\psi(q^8)\varphi(q^{24}) \\ &\quad\quad + 2q^6\varphi(q^{16})\psi(q^{48}) + 4q^{10}\psi(q^{32})\psi(q^{48}) + 4q^7\psi(q^8)\psi(q^{48})\big) \\ &\quad\quad \times \big(\varphi(q^{24})^2 + 4q^{12}\psi(q^{48})^2 + 4q^6\psi(q^{24})^2 \\ &\quad\quad + 4q^3\varphi(q^{48})\psi(q^{24}) + 8q^{15}\psi(q^{24})\psi(q^{96})\big). \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^s)^r$ and $\psi(q^s)^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above and (1.8) we deduce that

$$\begin{aligned} &\sum_{n=0}^{\infty} N(1,3,3,6;8n+13) q^{8n+13} \\ &= \sum_{n=0}^{\infty} N(1,3,3,6;8n+5) q^{8n+5} \\ &= 2q\psi(q^8)\varphi(q^{24}) \cdot 4q^{12}\psi(q^{48})^2 + 2q^6\varphi(q^{16})\psi(q^{48}) \cdot 8q^{15}\psi(q^{24})\psi(q^{96}) \\ &\quad + 4q^{10}\psi(q^{32})\psi(q^{48}) \cdot 4q^3\psi(q^{24})\varphi(q^{48}) + 4q^7\psi(q^8)\psi(q^{48}) \cdot 4q^6\psi(q^{24})^2 \end{aligned}$$

and so

$$\begin{aligned} &\sum_{n=0}^{\infty} N(1,3,3,6;8n+13) q^{8n} \\ &= 8\psi(q^8)\psi(q^{24})^2\psi(q^{48}) + 16q^8\varphi(q^{16})\psi(q^{48})\psi(q^{24})\psi(q^{96}) \\ &\quad + 16\psi(q^{24})\psi(q^{32})\psi(q^{48})\varphi(q^{48}) + 16\psi(q^8)\psi(q^{48})\psi(q^{24})^2. \end{aligned}$$

By (1.11),

$$\psi(q^{32})\varphi(q^{48}) + q^8\varphi(q^{16})\psi(q^{96}) = \psi(q^8)\psi(q^{24}).$$

Hence,

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,3,3,6;8n+13)q^{8n}\\
&= 8\psi(q^8)\psi(q^{24})^2\psi(q^{48}) + 16\psi(q^{24})\psi(q^{48})\psi(q^8)\psi(q^{24})\\
&\quad + 16\psi(q^8)\psi(q^{48})\psi(q^{24})^2\\
&= 40\psi(q^8)\psi(q^{24})^2\psi(q^{48}).
\end{aligned}$$

Replacing q with $q^{1/8}$ we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,3,3,6;8n+13)q^n &= 40\psi(q)\psi(q^3)^2\psi(q^6)\\
&= 40\sum_{n=0}^{\infty} t'(1,3,3,6;n)q^n = \frac{5}{2}\sum_{n=0}^{\infty} t(1,3,3,6;n)q^n.
\end{aligned}$$

Hence

$$t(1,3,3,6;n) = \frac{2}{5}N(1,3,3,6;8n+13).$$

Now applying [9, Theorem 4.1] yields the remaining part.

Remark 2.2 Using Maple we find

$$t(a,b,c,d;n) = \frac{2}{5}N(a,b,c,d;8n+a+b+c+d)$$

for $(a,b,c,d) = (1,1,1,2),(1,1,1,3),(1,1,2,3),(1,1,3,9),(1,3,3,3),(1,3,3,6)$ and $(1,3,9,9)$. When $(a,b,c,d) = (1,1,1,2),(1,1,1,3)$ or $(1,1,2,3)$ the result follows from (1.4). The formula for $t(1,3,3,3;n)$ was given in [12, Theorem 5.5], and the cases $(1,1,3,9)$ and $(1,3,9,9)$ were solved by Wang and Sun in [15].

Theorem 2.11. *If $n \in \mathbb{N}$ and $n \equiv 1,2 \pmod 4$, then*

$$t(1,1,4,6;n) = 2N(1,1,4,6;2n+3).$$

Proof. By (1.12) and Lemma 2.4,

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,1,4,6;n)q^n &= \varphi(q)^2\varphi(q^4)\varphi(q^6)\\
&= \varphi(q)^2(\varphi(q^{16}) + 2q^4\psi(q^{32}))(\varphi(q^{24}) + 2q^6\psi(q^{48}))\\
&= \big(\varphi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q^2\psi(q^8)^2 + 4q\varphi(q^{16})\psi(q^8)\\
&\quad + 8q^5\psi(q^8)\psi(q^{32})\big)\big(\varphi(q^{16})\varphi(q^{24}) + 4q^{10}\psi(q^{32})\psi(q^{48})\\
&\quad + 2q^4\varphi(q^{24})\psi(q^{32}) + 2q^6\varphi(q^{16})\psi(q^{48})\big).
\end{aligned} \tag{2.4}$$

Thus,

$$\sum_{m=0}^{\infty} N(1,1,4,6;8m+5)q^{8m+5} = 16q^5\psi(q^8)\psi(q^{32})\varphi(q^{16})\varphi(q^{24}),$$

$$\sum_{m=0}^{\infty} N(1,1,4,6;8m+7)q^{8m+7}$$
$$= 8q^7\psi(q^8)\psi(q^{48})\big(\varphi(q^{16})^2 + 4q^8\psi(q^{32})^2\big).$$

From the above, (1.8) and (1.10) we deduce that

$$\begin{aligned}
&\sum_{m=0}^{\infty} N(1,1,4,6;8m+5)q^m \\
&= 16\psi(q)\psi(q^4)\varphi(q^2)\varphi(q^3) = 16\psi(q)\psi(q^2)^2\varphi(q^3), \\
&\sum_{m=0}^{\infty} N(1,1,4,6;8m+7)q^m \\
&= 8\psi(q)\psi(q^6)\big(\varphi(q^2)^2 + 4q\psi(q^4)^2\big) = 8\varphi(q)^2\psi(q)\psi(q^6).
\end{aligned} \tag{2.5}$$

On the other hand, from (2.2) we know that

$$\sum_{m=0}^{\infty} t'(1,1,4,6;4m+1)q^{4m+1} = 2q\psi(q^4)\psi(q^8)^2\varphi(q^{12}),$$
$$\sum_{m=0}^{\infty} t'(1,1,4,6;4m+2)q^{4m+2} = q^2\varphi(q^4)^2\psi(q^4)\psi(q^{24}).$$

Therefore,

$$\sum_{m=0}^{\infty} t'(1,1,4,6;4m+1)q^m = 2\psi(q)\psi(q^2)^2\varphi(q^3),$$
$$\sum_{m=0}^{\infty} t'(1,1,4,6;4m+2)q^m = \varphi(q)^2\psi(q)\psi(q^6).$$

This together with (2.5) yields $N(1,1,4,6;8m+5) = 8t'(1,1,4,6;4m+1)$ and $N(1,1,4,6;8m+7) = 8t'(1,1,4,6;4m+2)$. To complete the proof, we recall that $t(a,b,c,d;n) = 16t'(a,b,c,d;n)$.

Theorem 2.12. *For $n \in \mathbb{N}$ with $n \equiv 1 \pmod 4$ we have*

$$t(2,2,3,9;n) = \frac{4}{3}N(2,2,3,9;2n+4).$$

Proof. Using (1.8)-(1.12) we see that for $|q| < 1$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} t'(2,2,3,9;n)q^n \\
&= \psi(q^2)^2\psi(q^3)\psi(q^9) = \varphi(q^2)\psi(q^4)(\varphi(q^{18})\psi(q^{12}) + q^3\varphi(q^6)\psi(q^{36})) \\
&= (\varphi(q^8) + 2q^2\psi(q^{16}))\psi(q^4)\big((\varphi(q^{72}) + 2q^{18}\psi(q^{144}))\psi(q^{12}) \\
&\quad + q^3(\varphi(q^{24}) + 2q^6\psi(q^{48}))\psi(q^{36})\big).
\end{aligned}$$

Hence

$$\sum_{m=0}^{\infty} t'(2,2,3,9;4m+5)q^{4m+5}$$
$$= (2q^5\varphi(q^{24})\psi(q^{16}) + 2q^9\varphi(q^8)\psi(q^{48}))\psi(q^4)\psi(q^{36})$$
$$= 2q^5\psi(q^4)\psi(q^{12})\psi(q^4)\psi(q^{36})$$

and so

$$\sum_{m=0}^{\infty} t'(2,2,3,9;4m+5)q^m$$
$$= 2\psi(q)^2\psi(q^3)\psi(q^9) = 2\sum_{m=0}^{\infty} t'(1,1,3,9;m)q^m.$$

Therefore,

$$t(2,2,3,9;4m+5)$$
$$= 16t'(2,2,3,9;4m+5) = 32t'(1,1,3,9;m) = 2t(1,1,3,9;m).$$

Taking $a = 3$ and $k = 1$ in Theorem 2.3 we see that $t(1,1,3,9;m) = \frac{2}{3}N(2,2,3,9;8m+14)$. Thus, $t(2,2,3,9;4m+5) = 2t(1,1,3,9;m) = \frac{4}{3}N(2,2,3,9;8m+14)$. This yields the result.

Theorem 2.13. *For $n \in \mathbb{N}$ we have*

$$t(1,2,2,6;n) = \frac{1}{2}N(1,1,4,6;8n+11)$$

and

$$t(1,1,8,12;2n) = \frac{1}{2}N(1,1,8,12;16n+22).$$

Proof. From (2.4) and (1.8) we know that

$$\sum_{m=0}^{\infty} N(1,1,4,6;8m+3)q^{8m+3}$$
$$= 32q^{11}\varphi(q^{16})\psi(q^8)\psi(q^{32})\psi(q^{48}) = 32q^{11}\psi(q^8)\psi(q^{16})^2\psi(q^{48}).$$

Hence

$$\sum_{m=0}^{\infty} N(1,1,4,6;8m+3)q^m = 32q\psi(q)\psi(q^2)^2\psi(q^6)$$
$$= 32\sum_{n=0}^{\infty} t'(1,2,2,6;n)q^{n+1} = 2\sum_{n=0}^{\infty} t(1,2,2,6;n)q^{n+1}.$$

Now comparing the coefficients of q^{n+1} we obtain $N(1,1,4,6;8n+11) = 2t(1,2,2,6;n)$. Applying Lemma 2.1 and Theorem 2.7 we see that

$$t(1,1,8,12;2n)$$

$$= t(1,2,2,6;n) = \frac{1}{2}N(1,1,4,6;8n+11) = \frac{1}{2}N(1,1,8,12;16n+22).$$

We are done.

Theorem 2.14. *For $m \in \mathbb{N}$ we have*

$$t(1,1,6,24;4m+1) = 2t(2,2,3,3;m)$$

and

$$t(1,1,6,24;4m) = t(1,1,3,3;m) = 2^{\alpha+4}\sigma(m_1),$$

where α and m_1 are given by $m+1 = 2^\alpha 3^\beta m_1$ and $gcd(m_1,6) = 1$.

Proof. Taking $a = 1$ and $k = 6$ in (2.2) we see that

$$\begin{aligned}&\sum_{n=0}^{\infty} t'(1,1,6,24;n)q^n \\ &= \big(\varphi(q^4)\varphi(q^{12})\psi(q^8) + 2q\psi(q^8)^2\varphi(q^{12}) + q^2\varphi(q^4)^2\psi(q^{24}) \\ &\quad + 2q^3\varphi(q^4)\psi(q^8)\psi(q^{24})\big)\psi(q^{24}).\end{aligned}$$

Hence, using (1.8) we see that

$$\begin{aligned}&\sum_{m=0}^{\infty} t'(1,1,6,24;4m)q^{4m} \\ &= \varphi(q^4)\psi(q^8)\varphi(q^{12})\psi(q^{24}) = \psi(q^4)^2\psi(q^{12})^2\end{aligned}$$

and

$$\begin{aligned}&\sum_{m=0}^{\infty} t'(1,1,6,24;4m+1)q^{4m+1} \\ &= 2q\psi(q^8)^2\varphi(q^{12})\psi(q^{24}) = 2q\psi(q^8)^2\psi(q^{12})^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{m=0}^{\infty} t'(1,1,6,24;4m)q^m = \psi(q)^2\psi(q^3)^2 = \sum_{m=0}^{\infty} t'(1,1,3,3;m)q^m, \\ &\sum_{m=0}^{\infty} t'(1,1,6,24;4m+1)q^m \\ &\quad = 2\psi(q^2)^2\psi(q^3)^2 = 2\sum_{m=0}^{\infty} t'(2,2,3,3;m)q^m.\end{aligned}$$

Since $t(a,b,c,d;n) = 16t'(a,b,c,d;n)$, we obtain $t(1,1,6,24;4m+1) = 2t(2,2,3,3;m)$ and $t(1,1,6,24;4m) = t(1,1,3,3;m)$. Now applying [15, Lemma 4.1] we deduce the theorem.

Theorem 2.15. *Suppose $a,b,c,d \in \mathbb{N}$, $2 \nmid ab$, $a \equiv b \pmod 4$ and $c \equiv 4 \pmod 8$. Then*

$$t(a,b,c,d;n)$$

$$= N(a,b,c,d;8n+a+b+c+d) - N(a,b,c,4d;8n+a+b+c+d).$$

Proof. For $m \in \mathbb{N}$ we see that

$$\begin{aligned} &N(a,b,c,d;m) \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; m = ax^2+by^2+cz^2+dw^2,\ 2 \mid w\right\}\right| \\ &\quad + \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; m = ax^2+by^2+cz^2+dw^2,\ 2 \nmid w\right\}\right|. \end{aligned}$$

Thus,

$$\begin{aligned} &N(a,b,c,d;8n+a+b+c+d) - N(a,b,c,4d;8n+a+b+c+d) \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; 8n+a+b+c+d = ax^2+by^2+cz^2+dw^2,\right.\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\left. 2 \nmid w\right\}\right|. \end{aligned}$$

Suppose $8n+a+b+c+d = ax^2+by^2+cz^2+dw^2$ for some $x,y,z,w \in \mathbb{Z}$ and $2 \nmid w$. Then $ax^2+by^2+cz^2 \equiv a+b+c \pmod 8$. If $2 \mid z$, then $ax^2+by^2 \equiv ax^2+by^2+cz^2 \equiv a+b+c \equiv a+b+4 \pmod 8$. Since $2 \nmid ab$ and $a+b \equiv 2 \pmod 4$ we see that $2 \nmid xy$ and so $ax^2+by^2 \equiv a+b \pmod 8$. We get a contradiction. Hence $2 \nmid z$ and so $ax^2+by^2 \equiv a+b \pmod 8$. This implies that $2 \nmid xy$. Therefore,

$$\begin{aligned} &N(a,b,c,d;8n+a+b+c+d) - N(a,b,c,4d;8n+a+b+c+d) \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; 8n+a+b+c+d = ax^2+by^2 \right.\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left.\left. +cz^2+dw^2,\ 2 \nmid w\right\}\right| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; 8n+a+b+c+d = ax^2+by^2 \right.\right. \\ &\qquad\qquad\qquad\qquad\qquad \left.\left. +cz^2+dw^2,\ 2 \nmid xyzw\right\}\right| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; 8n+a+b+c+d = a(2x-1)^2 \right.\right. \\ &\qquad\qquad \left.\left. +b(2y-1)^2+c(2z-1)^2+d(2w-1)^2\right\}\right| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; 8n = a\cdot 4x(x-1)+b\cdot 4y(y-1) \right.\right. \\ &\qquad\qquad\qquad \left.\left. +c\cdot 4z(z-1)+d\cdot 4w(w-1)\right\}\right| \\ &= t(a,b,c,d;n). \end{aligned}$$

This proves the theorem.

Calculations with Maple suggest the following conjectures on $t(a,b,c,d;n)$ (for $a \le b \le c \le d$, $a \le 10$, $b \le 20$, $c \le 30$ and $d \le 40$).

Conjecture 2.1. *Let $n \in \mathbb{N}$ with $n \equiv 0,3 \pmod 4$. Then*

$$t(1,1,4,6;n) = \frac{2}{3}N(1,1,4,6;8n+12) - N(1,1,4,6;2n+3).$$

Conjecture 2.2. *Let $n \in \mathbb{N}$ with $3 \mid n$. Then*

$$t(1,1,8,12;n) = \frac{1}{2}N(1,1,8,12;8n+22).$$

Conjecture 2.3. *Let $n \in \mathbb{N}$ with $3 \mid n$. Then*

$$t(1,3,8,8;n) = \frac{1}{3}N(1,3,8,8;8n+20) - 2N(1,3,8,8;2n+5).$$

Conjecture 2.4. *Let $n \in \mathbb{N}$ with $n \equiv 0 \pmod 6$. Then*

$$t(1,2,3,8;n) = \frac{2}{3}N(1,2,3,8;8n+14) - 2N(1,2,3,8;4n+7).$$

Conjecture 2.5. *Let $n \in \mathbb{N}$ with $n \equiv 0,2 \pmod 8$. Then*

$$t(1,2,4,17;n) = 4N(1,2,4,17;n+3).$$

Conjecture 2.6. *Let $n \in \mathbb{N}$. If $n \equiv 2,3 \pmod 5$, then*

$$t(1,1,5,8;n) = \frac{1}{2}N(1,1,5,8;8n+15).$$

Conjecture 2.7. *Let $n \in \mathbb{N}$. If $n \equiv 0,3,4,6,7 \pmod 9$, then*

$$t(1,1,8,9;n) = \frac{1}{2}N(1,1,8,9;8n+19).$$

Conjecture 2.8. *Let $n \in \mathbb{N}$. If $n \equiv 0,4,7,8,9,10 \pmod{13}$, then*

$$t(1,1,8,13;n) = \frac{1}{2}N(1,1,8,13;8n+23).$$

Conjecture 2.9. *Let $n \in \mathbb{N}$. If $n \equiv 0,3,5,6,7 \pmod{11}$, then*

$$t(1,1,4,11;n) = \frac{1}{3}N(1,1,4,11;8n+17).$$

Conjecture 2.10. *Let $n \in \mathbb{N}$. If $n \equiv 0,1,2,4,7 \pmod{11}$, then*

$$t(1,1,2,22;n) = \frac{1}{3}N(1,1,2,22;8n+26).$$

Conjecture 2.11. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 3$. Then*

$$t(1,3,12,36;n) = \frac{1}{2}N(1,3,12,36;8n+52) - 2N(1,3,12,36;2n+13).$$

Conjecture 2.12. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 4$. Then*

$$t(3,5,20,32;n) = \frac{1}{2}N(3,5,20,32;8n+60) - 2N(3,5,20,32;2n+15).$$

Conjecture 2.13. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 4$. Then*

$$t(1,6,15,18;n) = \frac{2}{3}N(1,6,15,18;8n+40) - 2N(1,6,15,18;2n+10).$$

Conjecture 2.14. *Let* $n \in \mathbb{N}$ *with* $n \equiv 1 \pmod 3$. *Then*

$$t(1,6,18,27;n) = \frac{2}{3}N(1,6,18,27;8n+52) - 2N(1,6,18,27;2n+13).$$

Conjecture 2.15. *Let* $n \in \mathbb{N}$ *with* $n \equiv 1 \pmod 3$. *Then*

$$t(1,8,9,18;n) = \frac{2}{3}N(1,8,9,18;8n+36) - 2N(1,8,9,18;2n+9).$$

Conjecture 2.16. *Let* $n \in \mathbb{N}$ *with* $4 \mid n$. *Then*

$$t(1,7,10,30;n) = \frac{2}{3}N(1,7,10,30;8n+48) - 2N(1,7,10,30;2n+12).$$

Conjecture 2.17. *Let* $n \in \mathbb{N}$ *with* $n \equiv 3 \pmod 4$. *Then*

$$t(1,10,15,30;n) = \frac{2}{3}N(1,10,15,30;8n+56) - 2N(1,10,15,30;2n+14).$$

Conjecture 2.18. *Let* $n \in \mathbb{N}$ *with* $n \equiv 2 \pmod 8$. *Then*

$$t(1,7,28,28;n) = \frac{2}{3}N(1,7,28,28;8n+64) - 2N(1,7,28,28;2n+16).$$

Conjecture 2.19. *Let* $n \in \mathbb{N}$ *with* $n \equiv 8 \pmod 9$. *Then*

$$t(1,9,16,18;n) = \frac{2}{3}N(1,9,16,18;8n+44) - 2N(1,9,16,18;2n+11).$$

Conjecture 2.20. *Let* $n \in \mathbb{N}$ *with* $n \equiv 1,7 \pmod 9$. *Then*

$$t(1,9,18,24;n) = \frac{2}{3}N(1,9,18,24;8n+52) - 2N(1,9,18,24;2n+13).$$

Conjecture 2.21. *Let* $n \in \mathbb{N}$ *with* $n \equiv 1,4 \pmod 9$. *Then*

$$t(1,9,18,32;n) = \frac{2}{3}N(1,9,18,32;8n+60) - 2N(1,9,18,32;2n+15).$$

Conjecture 2.22. *Let* $n \in \mathbb{N}$ *with* $n \equiv 5 \pmod 9$. *Then*

$$t(1,9,18,40;n) = \frac{2}{3}N(1,9,18,40;8n+68) - 2N(1,9,18,40;2n+17).$$

Conjecture 2.23. *Let* $n \in \mathbb{N}$ *with* $n \equiv 2,5 \pmod 9$. *Then*

$$\begin{aligned} &t(1,10,27,30;n) \\ &= \frac{2}{3}N(1,10,27,30;8n+68) - 2N(1,10,27,30;2n+17). \end{aligned}$$

We remark that Conjectures 2.1-2.5 have been checked for $n \leq 300$, Conjectures 2.6-2.17 have been checked for $n \leq 200$, Conjecture 2.18 has been checked for $n \leq 400$, and Conjectures 2.19-2.23 have been checked for $n \leq 500$.

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