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List of the results in the paper

THE COMBINATORIAL SUM $\sum\limits_{\substack{k=0\\ k\equiv r \pmod m}}^{n}\binom nk$ AND ITS APPLICATIONS IN NUMBER THEORY II

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Notations.

Let $\mathbb{Z}$ be the set of integers, and $\mathbb{N}$ the set of positive integers. For $r \in \mathbb{Z}$ and $m, n \in \mathbb{N}$ let

$$T^n_{r(m)} = \sum_{\substack{k=0\\ k\equiv r \pmod m}}^{n} \binom{n}{k}.$$

For $a, b \in \mathbb{Z}$ the Lucas sequences $\{u_n(a,b)\}$ and $\{v_n(a,b)\}$ are defined by

$$u_0(a,b) = 0,\ u_1(a,b) = 1 \text{ and } u_{n+1}(a,b) = bu_n(a,b) - au_{n-1}(a,b)\ (n \ge 1)$$

and

$$v_0(a,b) = 2,\ v_1(a,b) = b \text{ and } v_{n+1}(a,b) = bv_n(a,b) - av_{n-1}(a,b)\ (n \ge 1).$$

So $u_n(-1,2)$ is the Pell sequence.

In the paper $[\cdot]$ denotes the greatest integer function, $(\frac{a}{p})$ denotes the Legendre symbol, and $q_p(a) = (a^{p-1}-1)/p$ means the Fermat quotient.

For $m, n \in \mathbb{N}$ and $k \in \mathbb{Z}$ define

$$\Delta_m(k,n) = \begin{cases} mT^n_{\frac n2 + k(m)} - 2^n & \text{if } 2 \nmid m, \\ mT^n_{[\frac n2]+k(m)} - 2^n & \text{if } 2 \mid m. \end{cases}$$

2.1 Formulas for $\Delta_{16}(r,p)$ and $\Delta_8(r,p)$.

Theorem 2.1. *Let $p > 0$ be an odd number, and let $\{A_n\}$ and $\{B_n\}$ be defined as below:*

$$A_0 = B_0 = B_1 = 0,\ A_1 = 1,\ A_{n+1} = 4A_n - 2A_{n-1} - 2B_{n-1},$$
$$B_{n+1} = 4B_n - A_{n-1} - 2B_{n-1} \quad (n = 1, 2, 3, \dots).$$

Then $\Delta_{16}(r,p)=\Delta_{16}(17-r,p)$, *and*

$$\begin{aligned}
\Delta_{16}(1,p)&=2^{(p+1)/2}+4u_{\frac{p+1}{2}}(2,4)-4u_{\frac{p-1}{2}}(2,4)+8(A_{\frac{p+1}{2}}-A_{\frac{p-1}{2}}-B_{\frac{p-1}{2}}),\\
\Delta_{16}(2,p)&=-2^{(p+1)/2}+4u_{\frac{p-1}{2}}(2,4)+8(A_{\frac{p-1}{2}}+B_{\frac{p-1}{2}}-B_{\frac{p+1}{2}}),\\
\Delta_{16}(3,p)&=-2^{(p+1)/2}-4u_{\frac{p-1}{2}}(2,4)-8(B_{\frac{p+1}{2}}-B_{\frac{p-1}{2}}),\\
\Delta_{16}(4,p)&=2^{(p+1)/2}-4u_{\frac{p+1}{2}}(2,4)+4u_{\frac{p-1}{2}}(2,4)-8B_{\frac{p-1}{2}},\\
\Delta_{16}(5,p)&=2^{(p+1)/2}-4u_{\frac{p+1}{2}}(2,4)+4u_{\frac{p-1}{2}}(2,4)+8B_{\frac{p-1}{2}},\\
\Delta_{16}(6,p)&=-2^{(p+1)/2}-4u_{\frac{p-1}{2}}(2,4)+8(B_{\frac{p+1}{2}}-B_{\frac{p-1}{2}}),\\
\Delta_{16}(7,p)&=-2^{(p+1)/2}+4u_{\frac{p-1}{2}}(2,4)-8(A_{\frac{p-1}{2}}+B_{\frac{p-1}{2}}-B_{\frac{p+1}{2}}),\\
\Delta_{16}(8,p)&=2^{(p+1)/2}+4u_{\frac{p+1}{2}}(2,4)-4u_{\frac{p-1}{2}}(2,4)-8(A_{\frac{p+1}{2}}-A_{\frac{p-1}{2}}-B_{\frac{p-1}{2}}).
\end{aligned}$$

Corollary 2.1. *Let* p *be an odd prime,* $\varepsilon_r=1$ *or* 0 *according as* $p\equiv\pm(2r-1)\pmod{32}$ *or not. Then*

$$\begin{aligned}
u_{\frac{p-1}{2}}(2+\sqrt{2},4)&\equiv(\varepsilon_2-\varepsilon_3+\varepsilon_6-\varepsilon_7)+(\varepsilon_5-\varepsilon_4)\sqrt{2}\pmod p,\\
u_{\frac{p+1}{2}}(2+\sqrt{2},4)&\equiv(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6-\varepsilon_7-\varepsilon_8)\\
&\qquad+(\varepsilon_5+\varepsilon_6-\varepsilon_3-\varepsilon_4)\sqrt{2}\pmod p.
\end{aligned}$$

Theorem 2.2. *Let* $p>0$ *be odd, and* $\Delta_8(r,p)=8T^p_{\frac{p-1}{2}+r(8)}-2^p$.

(i) *If* $p\equiv1\pmod 4$, *then*

$$\begin{aligned}
\Delta_8(0,p)=\Delta_8(1,p)&=2^{\frac{p+1}{2}}+2^{\frac{p+7}{4}}u_{\frac{p+1}{2}}(-1,2),\\
\Delta_8(2,p)=\Delta_8(7,p)&=-2^{\frac{p+1}{2}}+2^{\frac{p+7}{4}}u_{\frac{p-1}{2}}(-1,2),\\
\Delta_8(3,p)=\Delta_8(6,p)&=-2^{\frac{p+1}{2}}-2^{\frac{p+7}{4}}u_{\frac{p-1}{2}}(-1,2),\\
\Delta_8(4,p)=\Delta_8(5,p)&=2^{\frac{p+1}{2}}-2^{\frac{p+7}{4}}u_{\frac{p+1}{2}}(-1,2).
\end{aligned}$$

(ii) *If* $p\equiv3\pmod 4$, *then*

$$\begin{aligned}
\Delta_8(0,p)=\Delta_8(1,p)&=2^{\frac{p+1}{2}}+2^{\frac{p+1}{4}}v_{\frac{p+1}{2}}(-1,2),\\
\Delta_8(2,p)=\Delta_8(7,p)&=-2^{\frac{p+1}{2}}+2^{\frac{p+1}{4}}v_{\frac{p-1}{2}}(-1,2),\\
\Delta_8(3,p)=\Delta_8(6,p)&=-2^{\frac{p+1}{2}}-2^{\frac{p+1}{4}}v_{\frac{p-1}{2}}(-1,2),\\
\Delta_8(4,p)=\Delta_8(5,p)&=2^{\frac{p+1}{2}}-2^{\frac{p+1}{4}}v_{\frac{p+1}{2}}(-1,2).
\end{aligned}$$

If $2 \mid n$, we can get the formulas for $\Delta_{16}(r,n)$ and $\Delta_8(r,n)$ by using Theorems 2.1, 2.2 and the following facts:

$$\Delta_8(r,n) = \Delta_8(r,n-1) + \Delta_8(r+1,n-1),$$
$$\Delta_{16}(r,n) = \Delta_{16}(r,n-1) + \Delta_{16}(r+1,n-1).$$

2.2 Some lemmas.

Lemma 2.1. *Suppose $p > 0$ is odd, $S = T^p_{0(8)} - T^p_{4(8)}$ and $D = T^p_{2(8)} - T^p_{6(8)}$.*

(1) *If $p \equiv 1 \pmod 8$, then*

$$S = (-1)^{\frac{p-1}{8}} 2^{\frac{p-1}{4}} u_{\frac{p+1}{2}}(-1,2), \quad D = (-1)^{\frac{p-1}{8}} 2^{\frac{p-1}{4}} u_{\frac{p-1}{2}}(-1,2).$$

(2) *If $p \equiv 3 \pmod 8$, then*

$$S = (-1)^{\frac{p-3}{8}} 2^{\frac{p-7}{4}} v_{\frac{p-1}{2}}(-1,2), \quad D = (-1)^{\frac{p-3}{8}} 2^{\frac{p-7}{4}} v_{\frac{p+1}{2}}(-1,2).$$

(3) *If $p \equiv 5 \pmod 8$, then*

$$S = (-1)^{\frac{p+3}{8}} 2^{\frac{p-1}{4}} u_{\frac{p-1}{2}}(-1,2), \quad D = (-1)^{\frac{p-5}{8}} 2^{\frac{p-1}{4}} u_{\frac{p+1}{2}}(-1,2).$$

(4) *If $p \equiv 7 \pmod 8$, then*

$$S = (-1)^{\frac{p+1}{8}} 2^{\frac{p-7}{4}} v_{\frac{p+1}{2}}(-1,2), \quad D = (-1)^{\frac{p-7}{8}} 2^{\frac{p-7}{4}} v_{\frac{p-1}{2}}(-1,2).$$

Lemma 2.2. *Let p be an odd prime, $S = T^p_{0(8)} - T^p_{4(8)}$ and $D = T^p_{2(8)} - T^p_{6(8)}$. Then*

$$S \equiv 1 + \frac{p}{4} \sum_{k=1}^{[\frac{p-1}{4}]} \frac{(-1)^{k-1}}{k} \pmod{p^2}, \quad D \equiv \frac{p}{2} \sum_{k=1}^{[\frac{p+1}{4}]} \frac{(-1)^k}{2k-1} \pmod{p^2}.$$

Lemma 2.3. *Let p be a prime of the form $4k+1$, and $b \in \mathbb{Z}$ with $(\frac{b^2+4}{p}) = 1$. Then*

$$p \mid u_{\frac{p-1}{4}}(-1,b) \quad \text{if and only if} \quad u_{\frac{p+1}{2}}(-1,b) \equiv (-1)^{\frac{p-1}{4}} \pmod p.$$

Lemma 2.4. *Let p be an odd prime, and $m \in \mathbb{Z}$ with $p \nmid m$. Then*

(1) $\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot m^k} \equiv \frac{m^{p-1}-1}{p} - 2 \cdot \frac{(\frac{m}{p})u_p(1-m,2)-1}{p} \pmod p$;

(2) $\sum_{k=1}^{(p-1)/2} \frac{m^k}{k} \equiv \frac{2-v_p(1-m,2)}{p} \pmod p$.

Lemma 2.5. *Let p be an odd prime, and $m \in \mathbb{Z}$ with $p \nmid m$. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{m^k}{k} - m \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot m^k} \equiv (m-1)q_p(m-1) - mq_p(m) \pmod{p}.$$

Corollary 2.2. *Let p be an odd prime, and $m \in \mathbb{Z}$ with $p \nmid m$. Then*

$$v_p(1-m,2) - 2m(\frac{m}{p})u_p(1-m,2) + (m-1)^p + (m-1) \equiv 0 \pmod{p^2}.$$

2.3 Number theoretical applications for $\Delta_8(r,p)$.

Theorem 2.3. *Let p be an odd prime, and $u_n = u_n(-1,2)$.*

(1) *If $p \equiv 1 \pmod 8$, then*

$$u_{\frac{p-1}{2}} \equiv 0 \pmod{p} \quad \text{and} \quad u_{\frac{p+1}{2}} \equiv (-1)^{\frac{p-1}{8}} 2^{\frac{p-1}{4}} \pmod{p}.$$

(2) *If $p \equiv 3 \pmod 8$, then*

$$u_{\frac{p-1}{2}} \equiv (-1)^{\frac{p-3}{8}} 2^{\frac{p-3}{4}} \pmod{p} \quad \text{and} \quad u_{\frac{p+1}{2}} \equiv (-1)^{\frac{p+5}{8}} 2^{\frac{p-3}{4}} \pmod{p}.$$

(3) *If $p \equiv 5 \pmod 8$, then*

$$u_{\frac{p-1}{2}} \equiv (-1)^{\frac{p-5}{8}} 2^{\frac{p-1}{4}} \pmod{p} \quad \text{and} \quad u_{\frac{p+1}{2}} \equiv 0 \pmod{p}.$$

(4) *If $p \equiv 7 \pmod 8$, then*

$$u_{\frac{p-1}{2}} \equiv (-1)^{\frac{p+1}{8}} 2^{\frac{p-3}{4}} \pmod{p} \quad \text{and} \quad u_{\frac{p+1}{2}} \equiv (-1)^{\frac{p+1}{8}} 2^{\frac{p-3}{4}} \pmod{p}.$$

Theorem 2.4. *Let p be a prime of the form $8k+1$, and hence $p = x^2 + 2y^2$ for some $x, y \in \mathbb{Z}$. Then $p \mid u_{\frac{p-1}{4}}(-1,2)$ if and only if $4 \mid y$.*

Let p be a prime of the form $8k+1$. Then $p = a^2 + b^2$ with $2 \nmid a$ and $4 \mid b$. It is well known that

$$2^{\frac{p-1}{4}} \equiv 1 \pmod{p} \iff 2 \text{ is a quartic residue of } p \iff 8 \mid b.$$

So, by using Lemma 2.3 and Theorem 2.3 we have

$$p \mid u_{\frac{p-1}{4}}(-1,2) \iff (-1)^{\frac{p-1}{8}} 2^{\frac{p-1}{4}} \equiv 1 \pmod{p}$$
$$\iff (-1)^{\frac{p-1}{8} + \frac{b}{4}} = 1 \iff \frac{p-1}{8} \equiv \frac{b}{4} \pmod{2}.$$

Theorem 2.5. *Let p be an odd prime. Then*

$$\frac{u_{p-(\frac{2}{p})}(-1,2)}{p} \equiv (-1)^{\frac{p-1}{2}} \sum_{k=1}^{[\frac{p+1}{4}]} \frac{(-1)^k}{2k-1} \pmod p.$$

Theorem 2.6. *Let p be an odd prime. Then*

$$\text{(i)} \quad \sum_{k=1}^{\frac{p-1}{2}} \frac{2^k}{k} \equiv 4(-1)^{\frac{p-1}{2}} \sum_{k=1}^{[\frac{p+1}{4}]} \frac{(-1)^{k-1}}{2k-1} \pmod p,$$

$$\text{(ii)} \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k\cdot 2^k} \equiv -4 \sum_{k=(1+(-1)^{\frac{p-1}{2}})/2}^{[p/8]} \frac{1}{4k-(-1)^{\frac{p-1}{2}}} \pmod p.$$

2.4 The formula for $\Delta_9(r,n)$.

Theorem 2.7. *For $n \in \mathbb{N}$ let $\Delta_9(r,n) = 9T^n_{\frac{n}{2}+r(9)} - 2^n$, and let*

$$F(0)=1,\ F(1)=0,\ F(2)=2,\ F(m+3)=3F(m+1)-F(m) \quad (m=0,1,2,\dots).$$

Then

$$\begin{aligned}
\Delta_9(0,n) &= 6F(n)+2(-1)^n,\\
\Delta_9(\pm 1,n) &= 3F(n)-3F(n-1)-(-1)^n,\\
\Delta_9(\pm 2,n) &= 3F(n-1)-3F(n)-3F(n+1)-(-1)^n,\\
\Delta_9(\pm 3,n) &= -3F(n)+2(-1)^n,\\
\Delta_9(\pm 4,n) &= 3F(n+1)-(-1)^n.
\end{aligned}$$

Corollary 2.3. *Let $p>3$ be a prime, and let $F(n)$ be defined in Theorem 2.7.*

(1) *If $p \equiv \pm 1 \pmod 9$, then*

$$F(p-1)\equiv 1 \pmod p, \quad F(p)\equiv 0 \pmod p, \quad F(p+1)\equiv 2 \pmod p.$$

(2) *If $p \equiv \pm 2 \pmod 9$, then*

$$F(p-1)\equiv -2 \pmod p, \quad F(p)\equiv 0 \pmod p, \quad F(p+1)\equiv -1 \pmod p.$$

(3) *If $p \equiv \pm 4 \pmod 9$, then*

$$F(p-1)\equiv 1 \pmod p, \quad F(p)\equiv 0 \pmod p, \quad F(p+1)\equiv -1 \pmod p.$$

Corollary 2.4. *Let $p > 3$ be a prime, and let $F(n)$ be defined in Theorem 2.7.*

(1) *If* $p \equiv \pm 1 \pmod 9$, *then*

$$\sum_{k=1}^{[p/9]} \frac{(-1)^{k-1}}{k} \equiv \frac{2^p - 2}{p} + 3 \cdot \frac{F(p+1) - 2}{p} \pmod p.$$

(2) *If* $p \equiv \pm 2 \pmod 9$, *then*

$$\sum_{k=1}^{[p/9]} \frac{(-1)^{k-1}}{k} \equiv \frac{2^p - 2}{p} + 3 \cdot \frac{F(p) - F(p-1) - 2}{p} \pmod p.$$

(3) *If* $p \equiv \pm 4 \pmod 9$, *then*

$$\sum_{k=1}^{[p/9]} \frac{(-1)^{k-1}}{k} \equiv \frac{2^p - 2}{p} + 3 \cdot \frac{F(p-1) - F(p) - F(p+1) - 2}{p} \pmod p.$$