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New supercongruences involving products of two binomial coefficients

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Abstract

Let $p>3$ be a prime, and let a be a rational p-adic integer with $a\not\equiv 0\pmod p$. We evaluate

$$\sum_{k=1}^{(p-1)/2}\frac 1k\binom ak\binom{-1-a}k \quad\text{and}\quad \sum_{k=0}^{(p-1)/2}\frac 1{2k-1}\binom ak\binom{-1-a}k$$

modulo p^2 in terms of Bernoulli and Euler polynomials.

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1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\ \sum_{k=0}^{n-1}\binom nk B_k=0\ (n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom nk B_kx^{n-k}\ (n\ge 0).$$

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by $E_0=1$,

$$E_n=-\sum_{k=1}^{[n/2]}\binom n{2k}E_{n-2k}\ (n\ge 1)\quad\text{and}\quad E_n(x)=\frac1{2^n}\sum_{k=0}^{n}\binom nk(2x-1)^{n-k}E_k\ (n\ge 0),$$

where $[a]$ is the greatest integer not exceeding a. In [12] the author introduced the sequence $\{U_n\}$ given by

$$U_0=1\quad\text{and}\quad U_n=-2\sum_{k=1}^{[n/2]}\binom n{2k}U_{n-2k}\ (n\ge 1).$$

It is well known that $B_{2n+1}=0$ and $E_{2n-1}=U_{2n-1}=0$ for any positive integer n. The sequences $\{B_n\}$, $\{E_n\}$ and $\{U_n\}$ have many interesting properties and applications (see for example [4]).

It is easily seen, as in [13], that

$$\begin{aligned}&\binom{-\frac12}k^2=\frac{\binom{2k}k^2}{16^k},\ \binom{-\frac13}k\binom{-\frac23}k=\frac{\binom{2k}k\binom{3k}k}{27^k},\\ &\binom{-\frac14}k\binom{-\frac34}k=\frac{\binom{2k}k\binom{4k}{2k}}{64^k},\ \binom{-\frac16}k\binom{-\frac56}k=\frac{\binom{3k}k\binom{6k}{3k}}{432^k}.\end{aligned}\tag{1.1}$$

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In 2003, Rodriguez-Villegas [8] conjectured that for any prime $p > 3$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},$$

where $(\frac{a}{p})$ is the Legendre symbol. These congruences were later confirmed by Mortenson [6,7].

Let $\mathbb{Z}$ be the set of integers. For a prime p let $\mathbb{Z}_p$ denote the set of rational numbers whose denominator is not divisible by p. Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$ and $a \not\equiv 0 \pmod p$. In [14-16], the author obtained congruences modulo p^3 for the sums

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k},\quad \sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k-1},$$
$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1},\quad \sum_{k=1}^{p-1}\frac{1}{k}\binom{a}{k}\binom{-1-a}{k}$$

in terms of Bernoulli and Euler polynomials.

Recently, Mao and Sun [5] obtained congruences modulo p^2 for

$$\sum_{k=0}^{(p-1)/2}\binom{a}{k}\binom{-1-a}{k}\quad\text{and}\quad\sum_{k=0}^{(p-1)/2}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k+1}.$$

In particular, these lead to congruences modulo p^2 for the sums

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k},\quad \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{4k}{2k}}{(2k+1)64^k},\quad \sum_{k=0}^{(p-1)/2}\frac{\binom{3k}{k}\binom{6k}{3k}}{(2k+1)432^k}.$$

In Theorems 2.2 and 3.4 below, we establish congruences modulo p^2 for

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k}\binom{a}{k}\binom{-1-a}{k}\quad\text{and}\quad\sum_{k=0}^{(p-1)/2}\frac{1}{2k-1}\binom{a}{k}\binom{-1-a}{k}$$

and give similar applications in Theorems 2.3, 3.3 and 3.5 to those in [5].

Throughout this paper, $H_n = \sum_{k=1}^{n}\frac{1}{k}$ for any positive integer n and $H_0 = 0$. For an odd prime p and $a \in \mathbb{Z}$ with $p \nmid a$, set $q_p(a) = (a^{p-1}-1)/p$. For $a \in \mathbb{Z}_p$, let $\langle a\rangle_p \in \{0, 1, \ldots, p-1\}$ be given by $a \equiv \langle a\rangle_p \pmod p$.

2. Congruences for $\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k}\binom{a}{k}\binom{-1-a}{k} \pmod{p^2}$

For any positive integer n and variable a let

$$S_n(a)=\sum_{k=1}^{n}\frac{1}{k}\binom{a}{k}\binom{-1-a}{k}.$$

By [14, (2.1)], for $a \neq 0$,

$$S_n(a) - S_n(a-1) = \frac{2}{a}\binom{a-1}{n}\binom{-a-1}{n} - \frac{2}{a}. \tag{2.1}$$

Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$ and $t = (a - \langle a \rangle_p)/p$. From (2.1),

$$\begin{aligned}
&S_n(a) - S_n(a - \langle a \rangle_p) \\
&= \sum_{k=0}^{\langle a \rangle_p - 1} (S_n(a-k) - S_n(a-k-1)) = \sum_{k=0}^{\langle a \rangle_p - 1} \frac{2}{a-k}\Big\{\binom{a-k-1}{n}\binom{k-a-1}{n} - 1\Big\} \\
&= \sum_{k=0}^{\langle a \rangle_p - 1} \frac{2}{pt + \langle a \rangle_p - k}\Big\{\binom{pt + \langle a \rangle_p - k - 1}{n}\binom{-pt - (\langle a \rangle_p - k) - 1}{n} - 1\Big\}.
\end{aligned}$$

Hence

$$S_n(a) - S_n(pt) = \sum_{r=1}^{\langle a \rangle_p} \frac{2}{pt+r}\Big\{\binom{pt+r-1}{n}\binom{-pt-r-1}{n} - 1\Big\}. \tag{2.2}$$

Lemma 2.1. *Let $p > 3$ be a prime, $r \in \{1, 2, \ldots, p-1\}$ and $t \in \mathbb{Z}_p$. Then*

$$\begin{aligned}
&\binom{pt+r-1}{\frac{p-1}{2}}\binom{-pt-r-1}{\frac{p-1}{2}} \\
&\equiv \begin{cases} \dfrac{pt}{r} + \dfrac{p^2 t}{r}\Big(2q_p(2) + H_{\frac{p-1}{2}-r}\Big) - \dfrac{p^2t^2}{r^2} \pmod{p^3} & \text{if } r < \dfrac{p}{2}, \\[2ex] \dfrac{p(t+1)}{r} + \dfrac{p^2(t+1)}{r}\Big(2q_p(2) + H_{r-\frac{p+1}{2}}\Big) - \dfrac{p^2 t(t+1)}{r^2} \pmod{p^3} & \text{if } r > \dfrac{p}{2}. \end{cases}
\end{aligned}$$

Proof. We first assume $r < \frac{p}{2}$. Observing that $\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}$ and

$$\binom{p-1}{k} = \frac{(p-1)(p-2)\cdots(p-k)}{k!} \equiv (-1)^k(1 - pH_k) \pmod{p^2},$$

we see that

$$\begin{aligned}
&\Big(\frac{p-1}{2}!\Big)^2\binom{pt+r-1}{\frac{p-1}{2}}\binom{-pt-r-1}{\frac{p-1}{2}} \\
&= \Big(\frac{p-1}{2}!\Big)^2\binom{pt+r-1}{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}}\binom{pt+r+\frac{p-1}{2}}{\frac{p-1}{2}} \\
&= (-1)^{\frac{p-1}{2}}\frac{pt}{pt+r}(pt-1)\cdots\Big(pt - (\frac{p-1}{2} - r)\Big)(pt+1)\cdots\Big(pt + \frac{p-1}{2} + r\Big) \\
&= (-1)^{\frac{p-1}{2}}\frac{pt}{pt+r}((pt)^2 - 1^2)((pt)^2 - 2^2)\cdots\Big((pt)^2 - (\frac{p-1}{2} - r)^2\Big) \\
&\quad \times \prod_{s=0}^{r-1}\Big(pt + \frac{p-1}{2} - s\Big)\Big(pt + p - \Big(\frac{p-1}{2} - s\Big)\Big) \\
&\equiv (-1)^r\frac{pt}{pt+r}\Big(\frac{p-1}{2} - r\Big)!^2\prod_{s=0}^{r-1}\Big(\frac{p-1}{2} - s\Big)\Big(\frac{p-1}{2} + s + 1\Big)
\end{aligned}$$

$$= (-1)^r \frac{pt}{pt+r}\Big(\frac{p-1}{2}-r\Big)!\Big(\frac{p-1}{2}+r\Big)! = (-1)^r \frac{pt}{pt+r}\cdot \frac{(p-1)!}{\binom{p-1}{\frac{p-1}{2}-r}}$$
$$\equiv (-1)^r \frac{pt}{pt+r}\cdot \frac{(p-1)!}{(-1)^{\frac{p-1}{2}-r}(1-pH_{\frac{p-1}{2}-r})} \equiv (-1)^{\frac{p-1}{2}}(p-1)!\cdot pt\frac{(r-pt)(1+pH_{\frac{p-1}{2}-r})}{r^2}$$
$$= (-1)^{\frac{p-1}{2}}(p-1)!\cdot pt\Big(\frac{1}{r}-\frac{pt}{r^2}+\frac{p}{r}H_{\frac{p-1}{2}-r}\Big) \pmod{p^3}.$$

From [3,(49)],

$$\frac{(p-1)!}{\frac{p-1}{2}!^2} = \binom{p-1}{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}}\big(1-pH_{\frac{p-1}{2}}\big) \equiv (-1)^{\frac{p-1}{2}}(1+2pq_p(2)) \pmod{p^2}$$

and we deduce that

$$\binom{pt+r-1}{\frac{p-1}{2}}\binom{-pt-r-1}{\frac{p-1}{2}} \equiv (1+2pq_p(2))pt\Big(\frac{1}{r}-\frac{pt}{r^2}+\frac{pH_{\frac{p-1}{2}-r}}{r}\Big)$$
$$\equiv pt\Big(\frac{1}{r}+\frac{p(2q_p(2)+H_{\frac{p-1}{2}-r})}{r}-\frac{pt}{r^2}\Big) \pmod{p^3}.$$

This yields the result in the case $r < \frac{p}{2}$.

Now assume $r > \frac{p}{2}$. Set $t' = -t-1$ and $r' = p-r$. Then

$$\binom{pt+r-1}{\frac{p-1}{2}}\binom{-pt-r-1}{\frac{p-1}{2}} = \binom{p(t+1)-(p-r)-1}{\frac{p-1}{2}}\binom{-p(t+1)+p-r-1}{\frac{p-1}{2}}$$
$$= \binom{pt'+r'-1}{\frac{p-1}{2}}\binom{-pt'-r'-1}{\frac{p-1}{2}}.$$

Since $r' < \frac{p}{2}$,

$$\binom{pt+r-1}{\frac{p-1}{2}}\binom{-pt-r-1}{\frac{p-1}{2}}$$
$$= \binom{pt'+r'-1}{\frac{p-1}{2}}\binom{-pt'-r'-1}{\frac{p-1}{2}} \equiv \frac{pt'}{r'}+\frac{p^2t'}{r'}\Big(2q_p(2)+H_{\frac{p-1}{2}-r'}\Big)-\frac{p^2t'^2}{r'^2}$$
$$= \frac{p(t+1)}{r-p}+\frac{p^2(t+1)}{r-p}\Big(2q_p(2)+H_{r-\frac{p+1}{2}}\Big)-\frac{p^2(t+1)^2}{(p-r)^2}$$
$$\equiv \frac{p(t+1)(r+p)}{r^2}+\frac{p^2(t+1)}{r}\Big(2q_p(2)+H_{r-\frac{p+1}{2}}\Big)-\frac{p^2(t+1)^2}{r^2} \pmod{p^3}.$$

This yields the result in the case $r > \frac{p}{2}$. Hence the lemma is proved.

We remark that Lemma 2.1 improves [5, Lemma 3.1].

Theorem 2.2. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$ and $a \not\equiv 0 \pmod p$. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{a}{k}\binom{-1-a}{k}}{k} + 2\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{a}{k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \langle a\rangle_p \le \frac{p-1}{2}, \\ pB_{p-2}(-a) \pmod{p^2} & \text{if } \langle a\rangle_p > \frac{p-1}{2} \end{cases}$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{a}{k}$$
$$\equiv -\frac{B_{2p-2}(-a)-B_{2p-2}}{2p-2}+2\frac{B_{p-1}(-a)-B_{p-1}}{p-1}-\frac{a-\langle a\rangle_p}{2}B_{p-2}(-a) \pmod{p^2}.$$

Proof. Set $t=(a-\langle a\rangle_p)/p$. By [10, Theorem 5.2], $\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv 0 \pmod p$. Thus,

$$\begin{aligned}S_{\frac{p-1}{2}}(pt)&=\sum_{k=1}^{(p-1)/2}\frac{1}{k}\binom{pt}{k}\binom{-pt-1}{k}=\sum_{k=1}^{(p-1)/2}\frac{pt}{k^2}\binom{pt-1}{k-1}\binom{-pt-1}{k}\\&\equiv\sum_{k=1}^{(p-1)/2}\frac{pt}{k^2}\binom{-1}{k-1}\binom{-1}{k}=-pt\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv 0 \pmod{p^2}.\end{aligned}$$

For $1\le\langle a\rangle_p\le\frac{p-1}{2}$, from (2.2) and Lemma 2.1,

$$\begin{aligned}S_{\frac{p-1}{2}}(a)&=S_{\frac{p-1}{2}}(pt)+\sum_{r=1}^{\langle a\rangle_p}\frac{2(r-pt)}{r^2-p^2t^2}\left\{\binom{pt+r-1}{\frac{p-1}{2}}\binom{-pt-r-1}{\frac{p-1}{2}}-1\right\}\\&\equiv 2\sum_{r=1}^{\langle a\rangle_p}\frac{r-pt}{r^2}\Big(\frac{pt}{r}-1\Big)\equiv 2\sum_{r=1}^{\langle a\rangle_p}\Big(\frac{2pt}{r^2}-\frac{1}{r}\Big) \pmod{p^2}.\end{aligned}$$

For $\langle a\rangle_p>\frac{p-1}{2}$, from (2.2) and Lemma 2.1,

$$\begin{aligned}S_{\frac{p-1}{2}}(a)&=S_{\frac{p-1}{2}}(pt)+\sum_{r=1}^{\langle a\rangle_p}\frac{2(r-pt)}{r^2-p^2t^2}\left\{\binom{pt+r-1}{\frac{p-1}{2}}\binom{-pt-r-1}{\frac{p-1}{2}}-1\right\}\\&\equiv 2\sum_{r=1}^{(p-1)/2}\frac{r-pt}{r^2}\Big(\frac{pt}{r}-1\Big)+2\sum_{r=(p+1)/2}^{\langle a\rangle_p}\frac{r-pt}{r^2}\Big(\frac{p(t+1)}{r}-1\Big)\\&=2\sum_{r=1}^{\langle a\rangle_p}\frac{r-pt}{r^2}\Big(\frac{p(t+1)}{r}-1\Big)-2\sum_{r=1}^{(p-1)/2}\frac{r-pt}{r^2}\cdot\frac{p}{r}\\&\equiv 2\sum_{r=1}^{\langle a\rangle_p}\Big(\frac{2pt}{r^2}-\frac{1}{r}\Big)+2p\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}-2p\sum_{r=1}^{(p-1)/2}\frac{1}{r^2} \pmod{p^2}.\end{aligned}$$

By [10, Theorem 5.2], $\sum_{r=1}^{(p-1)/2}\frac{1}{r^2}\equiv 0 \pmod p$. By [14, (3.4)] and Fermat's little theorem, $\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}\equiv\sum_{r=1}^{\langle a\rangle_p}r^{p-3}\equiv\frac{1}{2}B_{p-2}(-a) \pmod p$. It follows that

$$S_{\frac{p-1}{2}}(a)-2\sum_{r=1}^{\langle a\rangle_p}\Big(\frac{2pt}{r^2}-\frac{1}{r}\Big)\equiv\begin{cases}0 \pmod{p^2} & \text{if } \langle a\rangle_p\le\dfrac{p-1}{2},\\ pB_{p-2}(-a) \pmod{p^2} & \text{if } \langle a\rangle_p>\dfrac{p-1}{2}.\end{cases} \tag{2.3}$$

From [14, page 2402] and the fact that $\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0 \pmod p$ (see [10, Theorem 5.1]),

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{a}{k}\equiv -pt\sum_{k=1}^{p-1}\frac{1}{k^2}-\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}+2pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}\equiv\sum_{r=1}^{\langle a\rangle_p}\Big(\frac{2pt}{r^2}-\frac{1}{r}\Big) \pmod{p^2}.$$

This together with (2.3) yields the the first part.

Putting $b=p-1$ and $x=-a$ in [10, Theorem 3.1] and then applying [10, Theorem 2.2 (with $k=p^2-1$ and $n=2$)] we deduce that

$$\begin{aligned}\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)} &= \frac{B_{(p^2-1)(p-1)+p-1}(-a)-B_{(p^2-1)(p-1)+p-1}}{(p^2-1)(p-1)+p-1} \\ &\equiv (p^2-1)\frac{B_{p-1+p-1}(-a)-B_{p-1+p-1}}{p-1+p-1} - (p^2-2)\frac{B_{p-1}(-a)-B_{p-1}}{p-1} \\ &\equiv -\frac{B_{2p-2}(-a)-B_{2p-2}}{2p-2} + 2\frac{B_{p-1}(-a)-B_{p-1}}{p-1} \pmod{p^2}.\end{aligned}$$

This together with [14, Theorem 3.1] yields the remaining part. The proof is now complete.

Theorem 2.3. *Let $p>3$ be a prime. Then*

(i) $$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}\binom{4k}{2k}}{k\cdot 64^k} \equiv 6q_p(2) - p\big(3q_p(2)^2 + 2(-1)^{\frac{p-1}{2}}E_{p-3}\big) \pmod{p^2},$$

(ii) $$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}}{k\cdot 27^k} \equiv 3q_p(3) - p\Big(\frac{3}{2}q_p(3)^2 + 2\Big(\frac{p}{3}\Big)U_{p-3}\Big) \pmod{p^2},$$

(iii) $$\sum_{k=1}^{(p-1)/2}\frac{\binom{3k}{k}\binom{6k}{3k}}{k\cdot 432^k} \equiv 4q_p(2) + 3q_p(3) - p\Big(2q_p(2)^2 + \frac{3}{2}q_p(3)^2 + 5\Big(\frac{p}{3}\Big)U_{p-3}\Big) \pmod{p^2}.$$

Proof. From [3] or [11, (7.1)], $E_{2n} = -4^{2n+1}\frac{B_{2n+1}(\frac14)}{2n+1}$. Consequently, $E_{p-3} = -4^{p-2}\frac{B_{p-2}(\frac14)}{p-2} \equiv \frac18 B_{p-2}(\frac14) \pmod p$. Now, taking $a=-\frac14$ in Theorem 2.2,

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{\binom{-1/4}{k}\binom{-3/4}{k}}{k} + 2\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/4}{k} \\ &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4, \\ pB_{p-2}(\frac14) \equiv 8pE_{p-3} \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

By [14, Theorem 3.2],

$$-2\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/4}{k} \equiv 6q_p(2) + p(-3q_p(2)^2 - 2(2-(-1)^{\frac{p-1}{2}})E_{p-3}) \pmod{p^2}.$$

Hence, (i) follows by appealing to (1.1).

Taking $a=-\frac13$ in Theorem 2.2 and applying the fact that $B_{p-2}(\frac13)\equiv 6U_{p-3} \pmod p$ (see [12, page 217]), we see that

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{\binom{-1/3}{k}\binom{-2/3}{k}}{k} + 2\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/3}{k} \\ &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ pB_{p-2}(\frac13) \equiv 6pU_{p-3} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}\end{aligned}$$

By [14, Theorem 3.3],

$$-2\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/3}{k}\equiv 3q_p(3)-p\Big(\frac{3}{2}q_p(3)^2+\big(3-(\frac{p}{3})\big)U_{p-3}\Big) \pmod{p^2}.$$

Thus (ii) follows by appealing to (1.1).

Taking $a=-\frac{1}{6}$ in Theorem 2.2 and applying the fact $B_{p-2}(\frac{1}{6})\equiv 30U_{p-3} \pmod p$ (see [12, page 216]), we see that

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{\binom{-1/6}{k}\binom{-5/6}{k}}{k}+2\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/6}{k}\\ &\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ pB_{p-2}(\frac{1}{6})\equiv 30pU_{p-3} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}\end{aligned}$$

By [14, Theorem 3.4],

$$\begin{aligned}&-2\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/6}{k}\\ &\equiv 4q_p(2)+3q_p(3)-p\Big(2q_p(2)^2+\frac{3}{2}q_p(3)^2+5\big(3-2(\frac{p}{3})\big)U_{p-3}\Big) \pmod{p^2}.\end{aligned}$$

Thus, (iii) follows by appealing to (1.1).

Remark 2.4 Theorem 2.3(i) is equivalent to a conjecture made by Z.W. Sun (see [17, Conjecture 1.2]).

3. Congruences for $\sum_{k=0}^{\frac{p-1}{2}}\frac{1}{2k-1}\binom{a}{k}\binom{-1-a}{k} \pmod{p^2}$

We begin with a useful result from [5].

Lemma 3.1 ([5, page 254]). *Assume that $p>3$ is a prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0 \pmod p$ and $t=(a-\langle a\rangle_p)/p$. Then*

$$\sum_{k=0}^{(p-1)/2}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1}\equiv 1+2t+4ptq_p(2)+2pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}+2p\sum_{\substack{1\le r\le\langle a\rangle_p\\ r>\frac{p}{2}}}\frac{1}{r} \pmod{p^2}.$$

Lemma 3.2. *Let $p>3$ be a prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0 \pmod p$ and $t=(a-\langle a\rangle_p)/p$. Then*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1}\\ &\equiv\begin{cases}1+2t+4ptq_p(2)-2pt(B_{p-1}(-a)-B_{p-1}) \pmod{p^2} & \text{if } \langle a\rangle_p\le\frac{p-1}{2},\\ 1+2t+4p(t+1)q_p(2)-2p(t+1)(B_{p-1}(-a)-B_{p-1}) \pmod{p^2} & \text{if } \langle a\rangle_p>\frac{p-1}{2}.\end{cases}\end{aligned}$$

Proof. By [10, Lemma 3.2],

$$\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}\equiv\sum_{r=1}^{\langle a\rangle_p}r^{p-2}\equiv(-1)^{p-1}\frac{B_{p-1}(-a)-B_{p-1}}{p-1}+(-a+\langle a\rangle_p)B_{p-2}$$
$$\equiv-(B_{p-1}(-a)-B_{p-1})\pmod p. \tag{3.1}$$

Thus, for $\langle a\rangle_p\le\frac{p-1}{2}$ the result follows from Lemma 3.1.

Now assume that $\langle a\rangle_p>\frac{p-1}{2}$. Since $H_{\frac{p-1}{2}}\equiv-2q_p(2)\pmod p$,

$$\sum_{\substack{1\le r\le\langle a\rangle_p\\ r>\frac{p}{2}}}\frac{1}{r}=\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}-\sum_{r=1}^{(p-1)/2}\frac{1}{r}\equiv\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}+2q_p(2)\pmod p.$$

Hence the result follows from Lemma 3.1.

Let $p>3$ be a prime. From [1],

$$B_{p-1}\Big(\frac{1}{3}\Big)-B_{p-1}\equiv\frac{3}{2}q_p(3)\pmod p, \tag{3.2}$$

$$B_{p-1}\Big(\frac{1}{6}\Big)-B_{p-1}\equiv 2q_p(2)+\frac{3}{2}q_p(3)\pmod p. \tag{3.3}$$

Also, from [4],

$$E_n(x)=\frac{2}{n+1}\Big(B_{n+1}(x)-2^{n+1}B_{n+1}\Big(\frac{x}{2}\Big)\Big)=\frac{2^{n+1}}{n+1}\Big(B_{n+1}\Big(\frac{x+1}{2}\Big)-B_{n+1}\Big(\frac{x}{2}\Big)\Big). \tag{3.4}$$

If $a\in\mathbb{Z}_p$, then $B_{p-1}(a)-B_{p-1}\in\mathbb{Z}_p$ and $pB_{p-1}\equiv p-1\pmod p$ (see [2,9,10]). Thus,

(3.5)

$$\begin{aligned}E_{p-2}(-a)&=\frac{2}{p-1}\Big(B_{p-1}(-a)-2^{p-1}B_{p-1}\Big(-\frac{a}{2}\Big)\Big)\\&=\frac{2}{p-1}\Big(B_{p-1}(-a)-B_{p-1}-2^{p-1}\Big(B_{p-1}\Big(-\frac{a}{2}\Big)-B_{p-1}\Big)-\big(2^{p-1}-1\big)B_{p-1}\Big)\\&\equiv-2\Big(B_{p-1}(-a)-B_{p-1}-\Big(B_{p-1}\Big(-\frac{a}{2}\Big)-B_{p-1}\Big)+q_p(2)\Big)\pmod p.\end{aligned}$$

From (3.2), (3.3) and (3.5),

$$\begin{aligned}E_{p-2}\Big(\frac{1}{3}\Big)&\equiv-2\Big(B_{p-1}\Big(\frac{1}{3}\Big)-B_{p-1}-\Big(B_{p-1}\Big(\frac{1}{6}\Big)-B_{p-1}\Big)+q_p(2)\Big)\\&\equiv-2\Big(\frac{3}{2}q_p(3)-2q_p(2)-\frac{3}{2}q_p(3)+q_p(2)\Big)=2q_p(2)\pmod p.\end{aligned} \tag{3.6}$$

Theorem 3.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}}{(2k+1)27^k}\equiv\Big(\frac{p}{3}\Big)(2-2^{p+1}+3^p)\pmod{p^2}.$$

Proof. Taking $a=-\frac{1}{3}$ in Lemma 3.2 and noting that $t=-\frac{1}{3}$ or $-\frac{2}{3}$ according as $p\equiv 1 \pmod 3$ or $p\equiv 2 \pmod 3$, we see that

$$\sum_{k=0}^{(p-1)/2}\frac{1}{2k+1}\binom{-\frac{1}{3}}{k}\binom{-\frac{2}{3}}{k}\equiv\Big(\frac{p}{3}\Big)\Big(1-4pq_p(2)+2p\Big(B_{p-1}\Big(\frac{1}{3}\Big)-B_{p-1}\Big)\Big) \pmod{p^2}.$$

Now applying (3.2) yields

$$\sum_{k=0}^{(p-1)/2}\frac{1}{2k+1}\binom{-\frac{1}{3}}{k}\binom{-\frac{2}{3}}{k}\equiv\Big(\frac{p}{3}\Big)(1-4pq_p(2)+3pq_p(3))=\Big(\frac{p}{3}\Big)(2-2^{p+1}+3^p) \pmod{p^2}.$$

Together with (1.1), this gives the result.

Theorem 3.4 . *Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $a\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{(p-1)/2}\binom{a}{k}\binom{-1-a}{k}\frac{1}{2k-1}$$
$$\equiv\begin{cases}-(2a+1)+2(a-\langle a\rangle_p)(1+(2a+1)E_{p-2}(-2a)) \pmod{p^2} & \text{if } \langle a\rangle_p<\frac{p}{2},\\ 2a+1+2(p+a-\langle a\rangle_p)(1+(2a+1)E_{p-2}(-2a)) \pmod{p^2} & \text{if } \langle a\rangle_p>\frac{p}{2}.\end{cases}$$

Proof. By [15, Lemma 3.1],

$$\tag{3.7}\begin{aligned}&\sum_{k=0}^{n}\binom{a}{k}\binom{-1-a}{k}\frac{(2a(a+1)+1)k-a(a+1)}{4k^2-1}\\&=\frac{a(a+1)}{2n+1}\binom{a-1}{n}\binom{-2-a}{n}=\frac{a(a+n+1)}{2n+1}\binom{a-1}{n}\binom{-a-1}{n}.\end{aligned}$$

Set $t=(a-\langle a\rangle_p)/p$. Since $a\equiv\langle a\rangle_p \pmod p$, from Lemma 2.1,

$$\frac{1}{p}\binom{a-1}{\frac{p-1}{2}}\binom{-a-1}{\frac{p-1}{2}}$$
$$\equiv\begin{cases}\frac{t}{\langle a\rangle_p}+\frac{pt}{a}(2q_p(2)+H_{\frac{p-1}{2}-\langle a\rangle_p})-\frac{pt^2}{a^2} \pmod{p^2} & \text{if } \langle a\rangle_p<\frac{p}{2},\\ \frac{t+1}{\langle a\rangle_p}+\frac{p(t+1)}{a}(2q_p(2)+H_{\langle a\rangle_p-\frac{p+1}{2}})-\frac{pt(t+1)}{a^2} \pmod{p^2} & \text{if } \langle a\rangle_p>\frac{p}{2}.\end{cases}$$

For $\langle a\rangle_p<\frac{p}{2}$, using (3.1), [10, Lemma 3.1] and $B_n(1-x)=(-1)^nB_n(x)$ (see [4]),

$$\begin{aligned}H_{\frac{p-1}{2}-\langle a\rangle_p}&\equiv B_{p-1}-B_{p-1}\Big(\langle a\rangle_p-\frac{p-1}{2}\Big)\equiv B_{p-1}-B_{p-1}\Big(\frac{1}{2}+a\Big)\\&=B_{p-1}-B_{p-1}\Big(\frac{1}{2}-a\Big) \pmod p.\end{aligned}$$

For $\langle a\rangle_p>\frac{p}{2}$, using (3.1) and [10, Lemma 3.1],

$$H_{\langle a\rangle_p-\frac{p+1}{2}}\equiv B_{p-1}-B_{p-1}\Big(\frac{p+1}{2}-\langle a\rangle_p\Big)\equiv B_{p-1}-B_{p-1}\Big(\frac{1}{2}-a\Big) \pmod p.$$

Taking $n = \frac{p-1}{2}$ in (3.7) and applying these results,

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2} \binom{a}{k}\binom{-1-a}{k}\frac{(2a(a+1)+1)k - a(a+1)}{4k^2-1} \\
&= a\Big(a+\frac{p+1}{2}\Big)\frac{1}{p}\binom{a-1}{\frac{p-1}{2}}\binom{-a-1}{\frac{p-1}{2}} \\
&\equiv \begin{cases} (a+\frac{p+1}{2})\Big(\frac{t(pt+\langle a\rangle_p)}{\langle a\rangle_p} + pt\Big(2q_p(2)+B_{p-1}-B_{p-1}(\frac{1}{2}-a)\Big) - \frac{pt^2}{a}\Big) \pmod{p^2} \\ \qquad\qquad\qquad\qquad \text{if } \langle a\rangle_p < \frac{p}{2}, \\ (a+\frac{p+1}{2})\Big(\frac{(t+1)(pt+\langle a\rangle_p)}{\langle a\rangle_p} + p(t+1)\Big(2q_p(2)+B_{p-1}-B_{p-1}(\frac{1}{2}-a)\Big) \\ \qquad -\frac{pt(t+1)}{a}\Big) \pmod{p^2} \quad \text{if } \langle a\rangle_p > \frac{p}{2} \end{cases} \\
&\equiv t'\Big(\frac{p}{2}+\frac{2a+1}{2}\Big(1+p\big(2q_p(2)+B_{p-1}-B_{p-1}(\frac{1}{2}-a)\big)\Big) \pmod{p^2},
\end{aligned}$$

where $t' = t$ or $t+1$ according as $\langle a\rangle_p < \frac{p}{2}$ or $\langle a\rangle_p > \frac{p}{2}$. As

$$\frac{1}{2k-1} = 4\frac{(2a(a+1)+1)k - a(a+1)}{4k^2-1} - (2a+1)\frac{2a+1}{2k+1},$$

the preceding congruences and Lemma 3.2 yield

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2} \frac{1}{2k-1}\binom{a}{k}\binom{-1-a}{k} \\
&= 4\sum_{k=0}^{(p-1)/2} \binom{a}{k}\binom{-1-a}{k}\frac{(2a(a+1)+1)k - a(a+1)}{4k^2-1} \\
&\quad - (2a+1)\sum_{k=0}^{(p-1)/2} \binom{a}{k}\binom{-1-a}{k}\frac{2a+1}{2k+1} \\
&\equiv 2pt' + 2(2a+1)t'\Big(1+p\big(2q_p(2)+B_{p-1}-B_{p-1}(\frac{1}{2}-a)\big)\Big) \\
&\quad - (-1)^{t'-t}(2a+1) - 2(2a+1)t'\big(1+p\big(2q_p(2)-B_{p-1}(-a)+B_{p-1}\big)\big) \\
&= -(-1)^{t'-t}(2a+1) + 2pt' + 2(2a+1)pt'\Big(B_{p-1}(-a) - B_{p-1}(\frac{1}{2}-a)\Big) \pmod{p^2}.
\end{aligned}$$

By (3.4),

(3.8)

$$E_{p-2}(-2a) = \frac{2^{p-1}}{p-1}\Big(B_{p-1}\big(-a+\frac{1}{2}\big) - B_{p-1}(-a)\Big) \equiv B_{p-1}(-a) - B_{p-1}\Big(\frac{1}{2}-a\Big) \pmod{p}.$$

Now combining these congruences gives the result.

Theorem 3.5. *Let p be a prime with $p > 3$. Then*

(i) $$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{4k}{2k}}{(2k-1)64^k} \equiv (-1)^{\frac{p+1}{2}}\frac{p+1}{2} \pmod{p^2},$$

(ii) $$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}}{(2k-1)27^k}\equiv\frac{1}{9}\Big(\frac{p}{3}\Big)\big(2^{p+1}-7-6p\big)\pmod{p^2},$$

(iii) $$\sum_{k=0}^{(p-1)/2}\frac{\binom{3k}{k}\binom{6k}{3k}}{(2k-1)432^k}\equiv-\frac{1}{9}\Big(\frac{p}{3}\Big)\big(2^{p+1}+2+3p\big)\pmod{p^2}.$$

Proof. It is well known that $B_n(1-x)=(-1)^nB_n(x)$ and $E_n(1-x)=(-1)^nE_n(x)$[4]. For $m=3,4,6$, we see that $\langle-\frac{1}{m}\rangle_p=\frac{p-1}{m}$ or $\frac{(m-1)p-1}{m}$ according as $p\equiv1\pmod m$ or not. From (3.4), $E_{p-2}(\frac12)\equiv B_{p-1}(\frac14)-B_{p-1}(\frac34)\equiv0\pmod p$. Now taking $a=-\frac14$ in Theorem 3.4 and then applying (1.1) yields (i).

From (3.6), $E_{p-2}(\frac23)=-E_{p-2}(\frac13)\equiv-2q_p(2)\pmod p$. Taking $a=-\frac13$ in Theorem 3.4 and then applying (1.1) yields (ii). Taking $a=-\frac16$ in Theorem 3.4 and then applying (3.6) and (1.1) yields (iii). This completes the proof.

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